1.2. The Finite Calculus

The results in the previous sections are beautiful, but some of the proofs are almost *too* clever. In this section we will see some structure that simplifies things. This will build on skills you already have from studying calculus.

For example, if we want to go beyond triangular numbers and squares, the next step is pentagonal numbers. But the pictures are hard to draw because of the fivefold symmetry of the pentagon. Instead, consider what we've done so far:

$$\begin{array}{rrrrrrl} n\colon & 1 & 2 & 3 & 4 & 5 & \ldots, \\ t_n\colon & 1 & 3 & 6 & 10 & 15 & \ldots, \\ s_n\colon & 1 & 4 & 9 & 16 & 25 & \ldots. \end{array}$$

In each row, consider the differences between consecutive terms:

$$\begin{array}{rrrrrrl} (n+1)-n\colon & 1 & 1 & 1 & 1 & 1 & \ldots, \\ t_{n+1}-t_n\colon & 2 & 3 & 4 & 5 & 6 & \ldots, \\ s_{n+1}-s_n\colon & 3 & 5 & 7 & 9 & 11 & \ldots. \end{array}$$

There is nothing new here; in the third row, we are just seeing that each square is formed by adding an odd number (gnomon) to the previous square. If we now compute the differences again, we see

$$\begin{array}{rrrrrl} 0 & 0 & 0 & 0 & 0 & \ldots, \\ 1 & 1 & 1 & 1 & 1 & \ldots, \\ 2 & 2 & 2 & 2 & 2 & \ldots. \end{array}$$

In each case, the second differences are constant, and the constant increases by one in each row.

For convenience we will introduce the DIFFERENCE OPERATOR, Δ, on functions $f(n)$, which gives a new function, $\Delta f(n)$, defined as $f(n+1) - f(n)$. This is an analog of derivative. We can do it again,

$$\begin{aligned} \Delta^2 f(n) &= \Delta(\Delta f)(n) \\ &= (\Delta f)(n+1) - (\Delta f)(n) \\ &= f(n+2) - 2f(n+1) + f(n), \end{aligned}$$

in an analogy with the second derivative. Think of the triangular numbers and square numbers as functions and not sequences. So,

$$\begin{aligned}
s(n) &= n^2,\\
\Delta s(n) &= (n+1)^2 - n^2\\
&= n^2 + 2n + 1 - n^2 = 2n + 1,\\
\Delta^2 s(n) &= (2(n+1)+1) - (2n+1) = 2.
\end{aligned}$$

Based on the pattern of second differences, we expect that the pentagonal numbers, $p(n)$, should satisfy $\Delta^2 p(n) = 3$ for all n. This means that $\Delta p(n) = 3n + C$ for some constant C, since

$$\Delta(3n + C) = (3(n+1) + C) - (3n + C) = 3.$$

What about $p(n)$ itself? To correspond to the $+C$ term, we need a term, $Cn + D$ for some other constant D, since

$$\Delta(Cn + D) = (C(n+1) + D) - (Cn + D) = C.$$

We also need a term whose difference is $3n$. We already observed that for the triangular numbers, $\Delta t(n) = n + 1$. So, $\Delta t(n-1) = n$ and $\Delta(3t(n-1)) = 3n$. So,

$$p(n) = 3t(n-1) + Cn + D = 3(n-1)n/2 + Cn + D$$

for some constants C and D. We expect $p(1) = 1$ and $p(2) = 5$, because they are pentagonal numbers; so, plugging in, we get

$$\begin{aligned}
0 + C + D &= 1,\\
3 + 2C + D &= 5.
\end{aligned}$$

Solving, we get that $C = 1$ and $D = 0$, so

$$p(n) = 3(n-1)n/2 + n = n(3n-1)/2.$$

This seems to be correct, since it gives

$$\begin{array}{rcccccl}
p(n): & 1 & 5 & 12 & 22 & 35 & \ldots,\\
\Delta p(n): & 4 & 7 & 10 & 13 & 16 & \ldots,\\
\Delta^2 p(n): & 3 & 3 & 3 & 3 & 3 & \ldots.
\end{array}$$

Exercise 1.2.1. Imitate this argument to get a formula for the hexagonal numbers, $h(n)$.

The difference operator, Δ, has many similarities to the derivative d/dx in calculus. We have already used the fact that

$$\Delta(f+g)(n) = \Delta f(n) + \Delta g(n) \quad \text{and} \quad \Delta(c \cdot f)(n) = c \cdot \Delta f(n)$$

in an analogy with the corresponding rules for derivatives. But the rules are not exactly the same, since

$$\frac{d}{dx}x^2 = 2x \quad \text{but} \quad \Delta n^2 = 2n+1, \text{ not } 2n.$$

What functions play the role of powers x^m? It turns out to be the FACTORIAL POWERS

$$n^{\underline{m}} = \underbrace{n(n-1)(n-2)\cdots(n-(m-1))}_{m \text{ consecutive integers}}.$$

An empty product is 1 by convention, so

$$n^{\underline{0}} = 1, \quad n^{\underline{1}} = n, \quad n^{\underline{2}} = n(n-1), \quad n^{\underline{3}} = n(n-1)(n-2), \ldots. \tag{1.10}$$

Observe that

$$\begin{aligned} \Delta(n^{\underline{m}}) &= (n+1)^{\underline{m}} - n^{\underline{m}} \\ &= [(n+1)\cdots(n-(m-2))] - [n\cdots(n-(m-1))]. \end{aligned}$$

The last $m-1$ factors in the first term and the first $m-1$ factors in the second term are both equal to $n^{\underline{m-1}}$. So we have

$$\begin{aligned} \Delta(n^{\underline{m}}) &= [(n+1)\cdot n^{\underline{m-1}}] - [n^{\underline{m-1}} \cdot (n-(m-1))] \\ &= \{(n+1)-(n-(m-1))\} \cdot n^{\underline{m-1}} \\ &= m \cdot n^{\underline{m-1}}. \end{aligned}$$

What about negative powers? From Eq. (1.10) we see that

$$n^{\underline{2}} = \frac{n^{\underline{3}}}{n-2}, \quad n^{\underline{1}} = \frac{n^{\underline{2}}}{n-1}, \quad n^{\underline{0}} = \frac{n^{\underline{1}}}{n-0}.$$

It makes sense to define the negative powers so that the pattern continues:

$$\begin{aligned} n^{\underline{-1}} &= \frac{n^{\underline{0}}}{n--1} = \frac{1}{n+1}, \\ n^{\underline{-2}} &= \frac{n^{\underline{-1}}}{n--2} = \frac{1}{(n+1)(n+2)}, \\ n^{\underline{-3}} &= \frac{n^{\underline{-2}}}{n--3} = \frac{1}{(n+1)(n+2)(n+3)}, \\ &\vdots \end{aligned}$$

One can show that for any m, positive or negative,

$$\Delta(n^{\underline{m}}) = m \cdot n^{\underline{m-1}}. \tag{1.11}$$

Exercise 1.2.2. Verify this in the case of $m = -2$. That is, show that $\Delta(n^{\underline{-2}}) = -2 \cdot n^{\underline{-3}}$.

The factorial powers combine in a way that is a little more complicated than ordinary powers. Instead of $x^{m+k} = x^m \cdot x^k$, we have that

$$n^{\underline{m+k}} = n^{\underline{m}}(n-m)^{\underline{k}} \quad \text{for all } m, k. \tag{1.12}$$

Exercise 1.2.3. Verify this for $m = 2$ and $k = -3$. That is, show that $n^{\underline{-1}} = n^{\underline{2}}(n-2)^{\underline{-3}}$.

The difference operator, Δ, is like the derivative d/dx, and so one might ask about the operation that undoes Δ the way an antiderivative undoes a derivative. This operation is denoted Σ:

$$\Sigma f(n) = F(n), \quad \text{if } F(n) \text{ is a function with } \Delta F(n) = f(n).$$

Don't be confused by the symbol Σ; we are not computing any sums. $\Sigma f(n)$ denotes a *function*, not a number. As in calculus, there is more than one possible choice for $\Sigma f(n)$. We can add a constant C to $F(n)$, because $\Delta(C) = C - C = 0$. Just as in calculus, the rule (1.11) implies that

$$\Sigma n^{\underline{m}} = \frac{n^{\underline{m+1}}}{m+1} + C \quad \text{for } m \neq -1. \tag{1.13}$$

Exercise 1.2.4. We were already undoing the difference operator in finding pentagonal and hexagonal numbers. Generalize this to polygonal numbers with a sides, for any a. That is, find a formula for a function $f(n)$ with

$$\Delta^2 f(n) = a - 2, \quad \text{with } f(1) = 1 \text{ and } f(2) = a.$$

In calculus, the point of antiderivatives is to compute definite integrals. Geometrically, this is the area under curves. The Fundamental Theorem of Calculus says that if

$$F(x) = \int f(x)dx, \quad \text{then} \quad \int_a^b f(x)dx = F(b) - F(a).$$

We will think about this more carefully in Interlude 1, but for now the important point is the finite analog. We can use the operator Σ on functions to compute actual sums.

Theorem (Fundamental Theorem of Finite Calculus, Part I). *If*

$$\Sigma f(n) = F(n), \quad \textit{then} \quad \sum_{a \le n < b} f(n) = F(b) - F(a).$$

Proof. The hypothesis $\Sigma f(n) = F(n)$ is just another way to say that $f(n) = \Delta F(n)$. The sum on the left is

$$\begin{aligned}
\sum_{a \le n < b} f(n) &= f(a) + f(a+1) + \cdots + f(b-2) + f(b-1) \\
&= \Delta F(a) + \Delta F(a+1) + \cdots + \Delta F(b-2) + \Delta F(b-1) \\
&= (F(a+1) - F(a)) + (F(a+2) - F(a+1)) + \cdots \\
&\qquad \cdots + (F(b-1) - F(b-2)) + (F(b) - F(b-1)) \\
&= -F(a) + F(b).
\end{aligned}$$

□

Notice that it does not matter which choice of constant C we pick, because $(F(b)+C) - (F(a)+C) = F(b) - F(a)$.

As an application, we can use the fact that $\Sigma n^{\underline{1}} = \frac{n^{\underline{2}}}{2}$ to say that

$$1 + 2 + \cdots + n = \sum_{0 \le k < n+1} k^{\underline{1}} = \frac{(n+1)^{\underline{2}}}{2} - \frac{0^{\underline{2}}}{2} = \frac{n(n+1)}{2}.$$

This is formula (1.6) for triangular numbers.

Here is another example. Because

$$n^{\underline{1}} + n^{\underline{2}} = n + n(n-1) = n^2,$$

we can say that

$$\Sigma n^2 = \Sigma(n^{\underline{1}} + n^{\underline{2}}) = \frac{n^{\underline{2}}}{2} + \frac{n^{\underline{3}}}{3}.$$

So,

$$\begin{aligned}
\sum_{0 \le k < n+1} k^2 &= \left(\frac{(n+1)^{\underline{2}}}{2} + \frac{(n+1)^{\underline{3}}}{3}\right) - \left(\frac{0^{\underline{2}}}{2} + \frac{0^{\underline{3}}}{3}\right) \\
&= \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

This is just Eq. (1.7) again.

Exercise 1.2.5. First, verify that

$$n^{\underline{1}} + 3n^{\underline{2}} + n^{\underline{3}} = n^3.$$

Now use this fact to find formulas for

$$\sum_{0 \le k < n+1} k^3.$$

Your answer should agree with formula (1.8).

In fact, one can do this for any exponent m. We will see that there are integers called STIRLING NUMBERS, $\left\{ {m \atop k} \right\}$, which allow you to write ordinary powers in terms of factorial powers:

$$n^m = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} n^{\underline{k}}. \tag{1.14}$$

In the preceding example, we saw that

$$\left\{ {2 \atop 0} \right\} = 0, \left\{ {2 \atop 1} \right\} = 1, \left\{ {2 \atop 2} \right\} = 1.$$

In the first part of Exercise 1.2.5, you verified that

$$\left\{ {3 \atop 0} \right\} = 0, \left\{ {3 \atop 1} \right\} = 1, \left\{ {3 \atop 2} \right\} = 3, \left\{ {3 \atop 3} \right\} = 1.$$

Exercise 1.2.6. Use the Stirling numbers

$$\left\{ {4 \atop 0} \right\} = 0, \left\{ {4 \atop 1} \right\} = 1, \left\{ {4 \atop 2} \right\} = 7, \left\{ {4 \atop 3} \right\} = 6, \left\{ {4 \atop 4} \right\} = 1$$

to show that

$$1^4 + 2^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30. \tag{1.15}$$

The Stirling numbers are sort of like the binomial coefficients $\binom{m}{k}$. Binomial coefficients are found in Pascal's triangle, which you have probably seen:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

The first and last entry in each row is always 1; the rest are computed by adding the two binomial coefficients on either side in the previous row. Suppose we make a similar triangle for the Stirling numbers. The Stirling number $\left\{ {m \atop k} \right\}$ is the kth entry in row m here:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 7 & & 6 & & 1 & \\ 1 & & 15 & & 25 & & 10 & & 1 \end{array}$$

Exercise 1.2.7. Try to find the pattern in this triangle, similar to Pascal's. Here's a hint, but don't read it unless you're really stuck. The 3 is computed from the 1 and the second entry, also a 1, above it. The 7 is computed from the 1 and the second entry, a 3, above it. The 6 is computed from the 3 and the third entry, a 1, above it. What is the pattern?

Fill in the next row of Stirling's triangle.

In fact, if we make this a little more precise, we can prove the theorem now. First, though, we need to define

$$\left\{ {m \atop 0} \right\} = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m > 0, \end{cases} \qquad \text{and } \left\{ {m \atop k} \right\} = 0, \quad \text{if } k > m \text{ or } k < 0.$$

Theorem. *If we now* define *the Stirling numbers by the recursion you discovered, that is,*

$$\left\{ {m \atop k} \right\} = k \left\{ {m-1 \atop k} \right\} + \left\{ {m-1 \atop k-1} \right\},$$

then Eq. (1.14) is true.

Notice that we have switched our point of view; the recursion is now the definition and the property (1.14) that we are interested in is a theorem. This is perfectly legal, as long as we make it clear that is what is happening. You may have indexed things slightly differently; make sure your recursion is equivalent to this one.

Proof. We can prove Eq. (1.14) by induction. The case of $m = 1$ is already done. From the boundary conditions ($k > m$ or $k < 0$) defined earlier, we can write (1.14) more easily as a sum over all k:

$$n^m = \sum_k \left\{ {m \atop k} \right\} n^{\underline{k}}.$$

The extra terms are 0. For the inductive step, we can assume that

$$n^{m-1} = \sum_k \left\{ {m-1 \atop k} \right\} n^{\underline{k}}$$

in order to prove (1.14). But

$$\sum_k \left\{ {m \atop k} \right\} n^{\underline{k}} = \sum_k \left(k \left\{ {m-1 \atop k} \right\} + \left\{ {m-1 \atop k-1} \right\}\right) n^{\underline{k}}$$

by the recursion for Stirling numbers. Thus,

$$\sum_k \left\{ {m \atop k} \right\} n^{\underline{k}} = \sum_k k \left\{ {m-1 \atop k} \right\} n^{\underline{k}} + \sum_k \left\{ {m-1 \atop k-1} \right\} n^{\underline{k}}.$$

We need to notice that Eq. (1.12) implies

$$n^{\underline{k+1}} = n \cdot n^{\underline{k}} - k \cdot n^{\underline{k}},$$

so that

$$k \cdot n^{\underline{k}} = n \cdot n^{\underline{k}} - n^{\underline{k+1}}.$$

Plug this in to see that

$$\sum_k \left\{ {m \atop k} \right\} n^{\underline{k}} = \sum_k n \cdot \left\{ {m-1 \atop k} \right\} n^{\underline{k}} - \sum_k \left\{ {m-1 \atop k} \right\} n^{\underline{k+1}} + \sum_k \left\{ {m-1 \atop k-1} \right\} n^{\underline{k}}.$$

The last two sums cancel; they are secretly equal since the factorial power is always one more than the lower parameter in the Stirling number. So,

$$\sum_k \left\{ {m \atop k} \right\} n^{\underline{k}} = n \cdot \sum_k \left\{ {m-1 \atop k} \right\} n^{\underline{k}} = n \cdot n^{m-1} = n^m$$

by the induction hypothesis. □

Exercise 1.2.8. You now know enough to compute sums of any mth power in closed form. Show that

$$1^5 + 2^5 + \cdots + n^5 = (2n^2 + 2n - 1)(n+1)^2 n^2/12. \tag{1.16}$$

You can find out more about Stirling numbers in Graham, Knuth, and Patashnik, 1994.

As with the polygonal numbers, once we have a closed-form expression, there seems to be nothing left to say. But notice that the rule (1.13) misses one case. There is no factorial power whose difference is $n^{\underline{-1}}$. In other words, $\Sigma n^{\underline{-1}}$ is not a factorial power. (This is the finite analog of the calculus fact that no power of x has derivative $1/x$.) So we make a definition instead, defining the nth HARMONIC NUMBER to be

$$H_n = \sum_{1 \le k \le n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{1.17}$$

Notice that after changing the variable slightly, we can also write

$$H_n = \sum_{0 \le k < n} \frac{1}{k+1}.$$

What is ΔH_n? We compute

$$\begin{aligned} H_{n+1} - H_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} = n^{\underline{-1}}. \end{aligned}$$

So, the Harmonic numbers are the finite analog of logarithms in that

$$\Delta H_n = n^{\underline{-1}}$$

is true. Harmonic numbers are interesting, as shown in Eq. (1.17), which provides a generalization of the formulas (1.6), (1.7), (1.8), (1.15), and (1.16). In some sense they are even more interesting, because there is no closed-form expression for them as for the formulas mentioned earlier.

Actually, we can do this same procedure for *any* $f(n)$, not just $n^{\underline{-1}}$.

Theorem (Fundamental Theorem of Finite Calculus, Part II). *If a new function $F(n)$ is defined by*

$$F(n) = \sum_{0 \le k < n} f(n) \quad \textit{for some} \quad f(n),$$

then

$$\Delta F(n) = f(n), \quad \textit{so} \quad F(n) = \Sigma f(n).$$

Proof. This proof is exactly the same as the proof for the Harmonic numbers. □

Exercise 1.2.9. Suppose that $f(n) = 2^n$ (ordinary exponent, not factorial). Show that $\Delta f(n) = f(n)$ and $f(0) = 1$. What function in calculus are we imitating? Use the Fundamental Theorem, Part I, to show that

$$1 + 2 + 2^2 + \cdots + 2^n = \sum_{0 \le k < n+1} 2^k = 2^{n+1} - 1.$$

Exercise 1.2.10. More generally, suppose that $f(n) = x^n$. Here $x \neq 1$ is a constant, and n is still the variable. Show that $\Delta f(n) = (x - 1) f(n)$, and therefore $\Sigma f(n) = f(n)/(x - 1)$. Use this to show that

$$1 + x + x^2 + \cdots + x^n = \sum_{0 \le k < n+1} x^k = \frac{x^{n+1} - 1}{x - 1}.$$

This sum is called the GEOMETRIC SERIES.

Exercise 1.2.11. The Rhind papyrus is the oldest known mathematical document: 14 sheets of papyrus from the fifteenth dynasty, or about 1700 B.C. Problem 79 says, “There are seven houses. Each house has seven cats. Each cat catches seven mice. Each mouse eats seven ears of spelt [a grain related to wheat]. Each ear of spelt produces seven hekats [a bulk measure]. What is the total of all of these?” Use the Geometric series to answer this, the oldest known mathematical puzzle.

Archimedes, too, knew of the Geometric series.

Archimedes (287–212 B.C.). Archimedes is better known for his beautiful theorems on area and volume in geometry than for his work in number theory. However, the Geometric series and other series, as we will see, are vital in number theory. Archimedes used the Geometric series in his work *Quadrature of the Parabola*. He approximated the area below a parabola using a collection of congruent triangles. The sum of the areas was a Geometric series. Archimedes’ works were not widely studied until the Byzantines wrote commentaries in the sixth century A.D. Thabit ibn Qurra wrote commentaries in the ninth century. From these texts, Archimedes’ work became known in the west. Nicole Oresme quoted at length from Archimedes, as did Leonardo of Pisa.

Accounts of his death by Livy, Plutarch, and others all more or less agree that he was killed by a Roman soldier in the sack of Syracuse (in Sicily) in 212 B.C., while he was doing some mathematics. His grave was marked by a cylinder circumscribing a sphere, to commemorate his theorem in solid geometry: that the ratio of the volumes is 3:2. Cicero, as *Quaestor* of Sicily in 75 B.C., described his search for the site (Cicero, 1928):

> I shall call up from the dust on which he drew his figures an obscure, insignificant person, Archimedes. I tracked out his grave . . . and found it enclosed all round and covered with brambles and thickets . . . . I noticed a small column rising a little above the bushes, on which there was a figure of a sphere and a cylinder . . . . Slaves were sent in with sickles and when a passage to the place was opened we approached the pedestal; the epigram was traceable with about half of the lines legible, as the latter portion was worn away.

Cicero goes on to add,

> Who in all the world, who enjoys merely some degree of communion with the Muses, . . . is there who would not choose to be the mathematician rather than the tyrant?

The most useful trick in calculus for finding antiderivatives is "u substitution." This does not translate very well to finite calculus, except for very simple changes of variables involving translation. That is, if $\Delta f(k) = g(k)$ and a is any constant, then $\Delta(f(k+a)) = g(k+a)$.

Exercise 1.2.12. Use this and the fact that $2(k-1)^{\underline{-2}} = 1/t_k$ to find the sum of the reciprocals of the first n triangular numbers

$$\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}.$$

Can you compute

$$\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_n},$$

the sum of the reciprocals of the first n tetrahedral numbers?

Toward the end of this book, we will need one more tool based on this finite analog of calculus. If you are just casually skimming, you may skip the rest of this chapter. In calculus, another useful method of finding antiderivatives is integration by parts. This is exactly the same thing as the product rule for derivatives, just written in antiderivative notation. That is, if you have functions $u(x)$ and $v(x)$, then

$$(u(x)v(x))' = u(x)'v(x) + u(x)v(x)';$$

so,

$$u(x)v(x)' = (u(x)v(x))' - u(x)'v(x).$$

If we take antiderivatives of both sides of the equation and use the fact that $\int (u(x)v(x))'dx = u(x)v(x)$, we get

$$\int u(x)v(x)'dx = u(x)v(x) - \int u(x)'v(x)dx.$$

If we suppress mention of the variable x and use the abbreviations $u(x)'dx = du$ and $v(x)'dx = dv$, then this is the formula for integration by parts you know and love (or at least know):

$$\int udv = uv - \int vdu.$$

For a finite analog, it seems we should start by applying the difference operator, Δ, to a product of two functions, for an analog of the product rule. This gives

$$\Delta(u(n)v(n)) = u(n+1)v(n+1) - u(n)v(n).$$

We can add and subtract a term $u(n)v(n+1)$ to get

$$\begin{aligned}\Delta(u(n)v(n)) &= u(n+1)v(n+1) - u(n)v(n+1)\\ &\qquad + u(n)v(n+1) - u(n)v(n)\\ &= (u(n+1) - u(n))v(n+1)\\ &\qquad + u(n)(v(n+1) - v(n))\\ &= \Delta u(n)v(n+1) + u(n)\Delta v(n).\end{aligned}$$

This is not exactly what you might expect. The function v is shifted by one so that $v(n+1)$ appears. We will denote this shift operator on functions by E, so $Ef(n) = f(n+1)$. Then the product rule in this setting says

$$\Delta(uv) = \Delta u \cdot Ev + u \cdot \Delta v$$

when the variable n is suppressed. As in the derivation of the integration-by-parts formula, we rearrange the terms to say

$$u \cdot \Delta v = \Delta(uv) - \Delta u \cdot Ev.$$

Applying the Σ operator, which undoes Δ, we get that

$$\Sigma(u \cdot \Delta v) = uv - \Sigma(\Delta u \cdot Ev).$$

This identity is called SUMMATION BY PARTS. Remember that so far it is just an identity between functions.

Suppose we want to use Summation by Parts to compute

$$\sum_{0 \le k < n} k^{\underline{1}} H_k.$$

First we need to find the function $\Sigma(k^{\underline{1}} H_k)$. Let $u(k) = H_k$, so $\Delta u(k) = k^{\underline{-1}}$. Then $k^{\underline{1}} = \Delta v(k)$, so we can choose $v(k) = k^{\underline{2}}/2$. Summation by Parts says that

$$\Sigma(k^{\underline{1}} H_k) = H_k \cdot \frac{k^{\underline{2}}}{2} - \Sigma\left(E\left(\frac{k^{\underline{2}}}{2}\right) k^{\underline{-1}}\right).$$

Now $k^{\underline{2}}/2 = k(k-1)/2$, so $E(k^{\underline{2}}/2) = (k+1)k/2$, and then $E(k^{\underline{2}}/2)k^{\underline{-1}}$ is equal to $k/2 = k^{\underline{1}}/2$. Thus,

$$\begin{aligned} \Sigma k^{\underline{1}} H_k &= H_k \cdot \frac{k^{\underline{2}}}{2} - \Sigma \frac{k^{\underline{1}}}{2} \\ &= H_k \cdot \frac{k^{\underline{2}}}{2} - \frac{k^{\underline{2}}}{4} \\ &= \frac{k^{\underline{2}}}{2}\left(H_k - \frac{1}{2}\right). \end{aligned}$$

Remember, this is just saying that

$$\Delta\left(\frac{k^{\underline{2}}}{2}\left(H_k - \frac{1}{2}\right)\right) = k^{\underline{1}} H_k.$$

Now the Fundamental Theorem, Part I, says that

$$\begin{aligned} \sum_{0 \le k < n} k^{\underline{1}} H_k &= \left(\frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right)\right) - \left(\frac{0^{\underline{2}}}{2}\left(H_0 - \frac{1}{2}\right)\right) \\ &= \frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right). \end{aligned}$$

Exercise 1.2.13. Use Summation by Parts and the Fundamental Theorem to compute $\sum_{0 \le k < n} H_k$. (Hint: You can write $H_k = H_k \cdot 1 = H_k \cdot k^{\underline{0}}$.) Your answer will have Harmonic numbers in it, of course.

Exercise 1.2.14. Use Summation by Parts and the Fundamental Theorem to compute $\sum_{0 \le k < n} k2^k$. (Hint: You need the first part of Exercise 1.2.9.)