II.10 The Euler-Maclaurin Summation Formula

The King calls me "my Professor", and I am the happiest man in the world!
(Euler is proud to serve Frederick II in Berlin)

I have here a geometer who is a big cyclops ... who has only one eye left, and a new curve, which he is presently computing, could render him totally blind. (Frederick II; see Spiess 1929, p. 165-166.)

This formula was developed independently by Euler (1736) and Maclaurin (1742) as a powerful tool for the computation of sums such as the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, the sum of logarithms $\ln 2 + \ln 3 + \ln 4 + \ldots + \ln n = \ln n!$, the sum of powers $1^k + 2^k + 3^k + \ldots + n^k$, or the sum of reciprocal powers $1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots + \frac{1}{n^k}$, with the help of differential calculus.

Problem. For a given function $f(x)$, find a formula for

$$S = f(1) + f(2) + f(3) + \ldots + f(n) = \sum_{i=1}^{n} f(i) \tag{10.1}$$

("investigatio summae serierum ex termino generali").

Euler's Derivation of the Formula

The *first idea* (see Euler 1755, pars posterior, § 105, Maclaurin 1742, Book II, Chap. IV, p. 663f) is to consider also the sum with shifted arguments

$$s = f(0) + f(1) + f(2) + \ldots + f(n-1). \tag{10.2}$$

We compute the difference $S - s$ using Taylor's series (Eq. (2.8) with $x - x_0 = -1$)

$$f(i-1) - f(i) = -\frac{f'(i)}{1!} + \frac{f''(i)}{2!} - \frac{f'''(i)}{3!} + \ldots$$

and find

$$f(n) - f(0) = \sum_{i=1}^{n} f'(i) - \frac{1}{2!}\sum_{i=1}^{n} f''(i) + \frac{1}{3!}\sum_{i=1}^{n} f'''(i) - \frac{1}{4!}\sum_{i=1}^{n} f''''(i) + \ldots$$

In order to turn this formula for $\sum f'(i)$ into a formula for $\sum f(i)$, we replace f by its primitive (again denoted by f):

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2!}\sum_{i=1}^{n} f'(i) - \frac{1}{3!}\sum_{i=1}^{n} f''(i) + \frac{1}{4!}\sum_{i=1}^{n} f'''(i) - \ldots . \tag{10.3}$$

The *second idea* is to remove the sums $\sum f'$, $\sum f''$, $\sum f'''$, on the right by using the same formula, with f successively replaced by f', f'', f''' etc. This will lead to a formula of the type

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx - \alpha\big(f(n)-f(0)\big) + \beta\big(f'(n)-f'(0)\big) - \gamma\big(f''(n)-f''(0)\big) + \delta\big(f'''(n)-f'''(0)\big) - \ldots \tag{10.4}$$

For the computation of the coefficients $\alpha, \beta, \gamma, \ldots$ we successively replace f in (10.4) by $f', f'', \ldots$ to obtain

$$\begin{array}{llll} \sum f(i) & = \int_0^n f(x)\,dx & -\alpha(f(n)-f(0)) & +\beta(f'(n)-f'(0)) - \ldots \\ -\frac{1}{2!}\sum f'(i) & & = -\frac{1}{2!}(f(n)-f(0)) & +\frac{\alpha}{2!}(f'(n)-f'(0)) - \ldots \\ \frac{1}{3!}\sum f''(i) & & & = +\frac{1}{3!}(f'(n)-f'(0)) - \ldots \\ \vdots & & & \end{array}$$

The sum of all this, by (10.3), has to be $\int_0^n f(x)\,dx$. Therefore, we obtain

$$\alpha + \frac{1}{2!} = 0, \qquad \beta + \frac{\alpha}{2!} + \frac{1}{3!} = 0, \qquad \gamma + \frac{\beta}{2!} + \frac{\alpha}{3!} + \frac{1}{4!} = 0, \ldots , \tag{10.5}$$

from which we can compute $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{12}$, $\gamma = 0$, $\delta = -\frac{1}{720}, \ldots$ and we have

$$\boxed{\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2}\big(f(n)-f(0)\big) + \frac{1}{12}\,\big(f'(n)-f'(0)\big) - \frac{1}{720}\,\big(f'''(n)-f'''(0)\big) + \frac{1}{30240}\,\big(f^{(5)}(n)-f^{(5)}(0)\big) + \ldots .} \tag{10.6}$$

(10.1) Example. This formula, applied to a sum of nearly a million terms,

$$\frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \ldots + \frac{1}{1000000} = \ln(10^6) - \ln(10) + \frac{1}{2}\,10^{-6} - \frac{1}{20} + \frac{1}{1200} - \frac{1}{120}\,10^{-4} + \frac{1}{252}\,10^{-6} + \ldots \approx 11.463758469,$$

gives an excellent approximation of the exact result by a couple of terms only. The formula is, however, of no use for the computation of the first terms $1+\frac{1}{2}+\ldots+\frac{1}{10}$.

Bernoulli Numbers. It is customary to replace the coefficients α, β, $\gamma, \ldots$ by $B_i/i!$ ($B_0 = 1$, $\alpha = B_1/1!$, $\beta = B_2/2!, \ldots$), so that (10.5) becomes

$$2B_1 + B_0 = 0, \quad 3B_2 + 3B_1 + B_0 = 0, \quad \ldots , \quad \sum_{i=0}^{k-1} \binom{k}{i} B_i = 0. \tag{10.5'}$$

The Bernoulli numbers, as far as Euler calculated them, are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$
$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798},$$
$$B_{20} = -\frac{174611}{330}, \qquad B_{22} = \frac{854513}{138}, \qquad B_{24} = -\frac{236364091}{2730},$$
$$B_{26} = \frac{8553103}{6}, \qquad B_{28} = -\frac{23749461029}{870}, \qquad B_{30} = \frac{8615841276005}{14322},$$

and $B_3 = B_5 = \ldots = 0$. In this notation, Eq. (10.6) becomes

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2}\big(f(n) - f(0)\big) + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\Big). \tag{10.6'}$$

Example. For $f(x) = x^q$ the series of Eq. (10.6′) is finite and gives the well-known formula of Jac. Bernoulli (I.1.28), (I.1.29).

Generating Function. In order to get more insight into the Bernoulli numbers, we apply one of Euler's great ideas: consider the function $V(u)$ whose Taylor coefficients are the numbers under consideration, i.e., define

$$\begin{aligned} V(u) &= 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \ldots \\ &= 1 + \frac{B_1}{1!}u + \frac{B_2}{2!}u^2 + \frac{B_3}{3!}u^3 + \frac{B_4}{4!}u^4 + \ldots . \end{aligned} \tag{10.7}$$

Now the formulas (10.5) alias (10.5′) say simply that

$$V(u) \cdot \left(1 + \frac{u}{2!} + \frac{u^2}{3!} + \frac{u^3}{4!} + \ldots\right) = 1,$$

that is,

$$V(u) = \frac{u}{e^u - 1}. \tag{10.8}$$

Thus, the infinitely many *algebraic* equations become *one analytic* formula. The fact that

$$V(u) + \frac{u}{2} = \frac{u}{e^u - 1} + \frac{u}{2} = \frac{u}{2} \cdot \frac{e^{u/2} + e^{-u/2}}{e^{u/2} - e^{-u/2}} \tag{10.9}$$

is an *even* function shows that $B_3 = B_5 = B_7 = \ldots = 0$.

De Usu Legitimo Formulae Summatoriae Maclaurinianae

We now insert $f(x) = \cos(2\pi x)$, for which $f(i) = 1$ for all i, into Eq. (10.6′). This gives $1 + 1 + \ldots + 1$ to the left, and $0 + 0 + 0 + \ldots$ to the right, because $\cos(2\pi x)$ together with all its derivatives is periodic with period 1. We see that the formula as it stands *is wrong*! Another problem is that for most functions f the infinite series in (10.6′) usually does not converge.

It is therefore necessary to truncate the formula after a finite number of terms and to obtain an expression for the remainder. This was done in beautiful Latin (see above) by Jacobi (1834) by rearranging Euler's proof using the error term (4.32) of Bernoulli-Cauchy throughout. It was later discovered (Wirtinger 1902) that the proof can be done simply by repeated integration by parts in a similar manner to the proof of Eq. (4.32). The main ingredient of the proof is the so-called Bernoulli polynomials.

Bernoulli Polynomials. The polynomials

$$\begin{aligned}
B_1(x) &= B_0 x + B_1 && = x - \tfrac{1}{2} \\
B_2(x) &= B_0 x^2 + 2B_1 x + B_2 && = x^2 - x + \tfrac{1}{6} \\
B_3(x) &= B_0 x^3 + 3B_1 x^2 + 3B_2 x + B_3 && = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x \\
B_4(x) &= B_0 x^4 + 4B_1 x^3 + 6B_2 x^2 + 4B_3 x + B_4 && = x^4 - 2x^3 + x^2 - \tfrac{1}{30}\,,
\end{aligned}$$

or, in general,

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}, \tag{10.10}$$

satisfy

$$B_k'(x) = kB_{k-1}(x), \qquad B_k(0) = B_k(1) = B_k \qquad (k \geq 2). \tag{10.11}$$

Indeed, the first formula of (10.11) is a property of the binomial coefficients (see Theorem I.2.1); the second formula follows from the definition and from (10.5′).

(10.2) Theorem. *We have*

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2}\Big(f(n) - f(0)\Big)$$
$$+ \sum_{j=2}^{k} \frac{(-1)^j B_j}{j!} \Big(f^{(j-1)}(n) - f^{(j-1)}(0)\Big) + \widetilde{R}_k,$$

where

$$\widetilde{R}_k = \frac{(-1)^{k-1}}{k!} \int_0^n \widetilde{B}_k(x)\, f^{(k)}(x)\,dx. \tag{10.12}$$

Here, $\widetilde{B}_k(x)$ is equal to $B_k(x)$ for $0 \leq x \leq 1$ and extended periodically with period 1 (see Fig. 10.1).

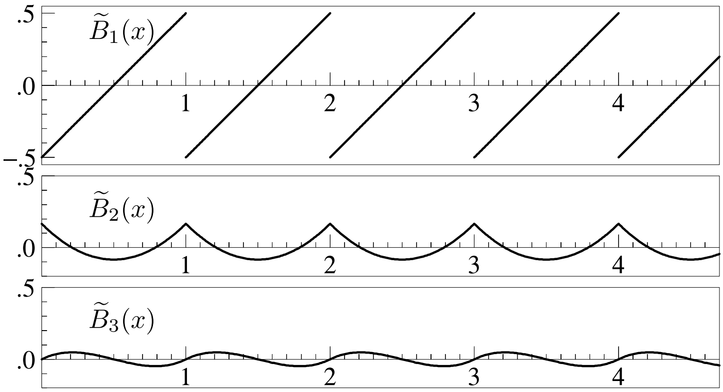

FIGURE 10.1. Bernoulli polynomials

Proof. We start by proving the statement for $n = 1$. Using $B_1'(x) = 1$ and integrating by parts we have

$$\int_0^1 f(x)\,dx = \int_0^1 B_1'(x)f(x)\,dx = B_1(x)f(x)\Big|_0^1 - \int_0^1 B_1(x)f'(x)\,dx.$$

The first term is $\frac{1}{2}(f(1) + f(0))$. In the second term we insert from (10.11) $B_1(x) = \frac{1}{2}B_2'(x)$ and integrate once again. This gives

$$\int_0^1 f(x)\,dx = \frac{1}{2}\Big(f(1)+f(0)\Big) - \frac{B_2}{2!}\Big(f'(1) - f'(0)\Big) + \frac{1}{2!}\int_0^1 B_2(x)f''(x)\,dx$$

or, continuing like this,

(10.13)

$$\frac{1}{2}\Big(f(1)+f(0)\Big) = \int_0^1 f(x)\,dx + \sum_{j=2}^{k} \frac{(-1)^j B_j}{j!}\Big(f^{(j-1)}(1) - f^{(j-1)}(0)\Big) + R_k,$$

with

(10.14) $$R_k = \frac{(-1)^{k-1}}{k!}\int_0^1 B_k(x)\,f^{(k)}(x)\,dx.$$

We next apply Eq. (10.14) to the shifted functions $f(x+i-1)$, observe that

$$\int_0^1 B_k(x)f^{(k)}(x+i-1)\,dx = \int_{i-1}^{i} \widetilde{B}_k(x)f^{(k)}(x)\,dx,$$

and obtain the statement of Theorem 10.2 by summing these formulas from $i = 1$ to $i = n$. □

Estimating the Remainder. The estimates (for $0 \le x \le 1$)

$$|B_1(x)| \le \frac{1}{2}, \qquad |B_2(x)| \le \frac{1}{6}, \qquad |B_3(x)| \le \frac{\sqrt{3}}{36}, \qquad |B_4(x)| \le \frac{1}{30},$$

which are easy to check, and the fact that $|\int_0^n g(x)\,dx| \le \int_0^n |g(x)|\,dx$, show that

$$|\widetilde{R}_1| \le \frac{1}{2}\int_0^n |f'(x)|\,dx, \quad |\widetilde{R}_2| \le \frac{1}{12}\int_0^n |f''(x)|\,dx, \quad \ldots . \tag{10.15}$$

These are the desired rigorous estimates of the remainder of Euler-Maclaurin's summation formula. Further maximal and minimal values of the Bernoulli polynomials have been computed by Lehmer (1940); see Exercise 10.3.

(10.3) *Remark.* If we apply the formula of Theorem 10.2 to the function $f(t) = hg(a+th)$ with $h = (b-a)/n$ and if we pass the term $\big(f(n)-f(0)\big)/2$ to the left side, we obtain (with $x_i = a + ih$)

$$\begin{aligned} \frac{h}{2}\,g(x_0) + h\sum_{i=1}^{n-1} g(x_i) + \frac{h}{2}\,g(x_n) = \int_a^b g(x)\,dx \\ + \sum_{j=2}^{k} \frac{h^j}{j!}\,B_j\Big(g^{(j-1)}(b) - g^{(j-1)}(a)\Big) \\ + (-1)^{k-1}\frac{h^{k+1}}{k!}\int_0^n \widetilde{B}_k(t) g^{(k)}(a+th)\,dt, \end{aligned} \tag{10.16}$$

where we recognize on the left the *trapezoidal rule*. Equation (10.16) shows that the dominating term of the error is $(h^2/12)\big(g'(b)-g'(a)\big)$. However, if g is periodic, then all terms in the Euler-Maclaurin series disappear and the error is equal to $\widetilde{R}_k$ for an arbitrary k; this explains the surprisingly good results of Table 6.2 (Sect. II.6).

Stirling's Formula

We put $f(x) = \ln x$ in the Euler-Maclaurin formula. Since

$$\sum_{i=2}^{n} f(i) = \ln 2 + \ln 3 + \ln 4 + \ln 5 + \ldots + \ln n = \ln(n!),$$

we will obtain an approximate expression for the factorials $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

(10.4) Theorem (Stirling 1730). *We have*

$$n! = \frac{\sqrt{2\pi n}\;n^n}{e^n}\cdot \exp\Big(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \widetilde{R}_9\Big), \tag{10.17}$$

where $|\widetilde{R}_9| \leq 0.0006605/n^8$. *This gives, for* $n \to \infty$, *the approximation*

$$n! \approx \frac{\sqrt{2\pi n}\ n^n}{e^n}. \tag{10.18}$$

Remark. This famous formula is especially useful in combinatorial analysis, statistics, and probability theory. Equation (10.17) is truncated after the 4th term simply because one additional term would not fit into the same line.

The numerical values of (10.18) and (10.17) (with one, two and three terms) for $n = 10$ and $n = 100$ are compared to $n!$ in Table 10.1.

TABLE 10.1. Factorial function and approximations by Stirling's formula

$n = 10:$	Stirling 0	$= 0.3598695618741035921623175932 83 \cdot 10^7$
	Stirling 1	$= 0.36288100514269335299411653167 5 \cdot 10^7$
	Stirling 2	$= 0.3628799971413012925385912239 41 \cdot 10^7$
	Stirling 3	$= 0.3628800002130128127907761286 2 \cdot 10^7$
	$n!$	$= 0.36288000000000000000000000000 \cdot 10^7$
$n = 100:$	Stirling 0	$= 0.93248476252693432477647561271 8 \cdot 10^{158}$
	Stirling 1	$= 0.93326215703176234098961919514 6 \cdot 10^{158}$
	Stirling 2	$= 0.93326215443936746394638335662 4 \cdot 10^{158}$
	Stirling 3	$= 0.93326215443944153237133886491 8 \cdot 10^{158}$
	$n!$	$= 0.93326215443944152681699238856 3 \cdot 10^{158}$

Proof. We have seen above (Example 10.1) that the Euler-Maclaurin formula is inefficient if the higher derivatives of $f(x)$ become large on the considered interval. We therefore apply the formula with $f(x) = \ln x$ for the sum from $i = n + 1$ to $i = m$. Since

$$\int \ln x\, dx = x \ln x - x, \qquad \frac{d^j}{dx^j}\big(\ln x\big) = (-1)^{j-1} \frac{(j-1)!}{x^j},$$

we obtain from Theorem 10.2 that

$$\begin{aligned}\sum_{i=n+1}^{m} f(i) = \ln m! - \ln n! &= m \ln m - m - (n \ln n - n) + \frac{1}{2}\big(\ln m - \ln n\big) \\ &+ \frac{1}{12}\Big(\frac{1}{m} - \frac{1}{n}\Big) - \frac{1}{360}\Big(\frac{1}{m^3} - \frac{1}{n^3}\Big) + \widetilde{R}_5,\end{aligned} \tag{10.19}$$

where $|\widetilde{R}_5| \leq 0.00123/n^4$ for all $m > n$. This estimate is obtained from (10.12) and (10.15) and the fact that $|B_5(x)| \leq 0.02446$ for $0 \leq x \leq 1$. In (10.19), the terms $\ln n!$, $n \ln n$, n, and $(1/2) \ln n$ diverge individually for $n \to \infty$. We therefore take them together and set

(10.20) $$\gamma_n = \ln n! + n - \Big(n + \frac{1}{2}\Big) \ln n,$$

and (10.19) becomes

(10.21) $$\gamma_n = \gamma_m + \frac{1}{12}\Big(\frac{1}{n} - \frac{1}{m}\Big) - \frac{1}{360}\Big(\frac{1}{n^3} - \frac{1}{m^3}\Big) - \widetilde{R}_5.$$

For n and m sufficiently large γ_n and γ_m become arbitrarily close. Therefore, it appears that the values γ_m converge, for $m \to \infty$, to a value that we denote by γ (the precise proof will be given in Theorem III.1.8 of Cauchy). We then take the limit $m \to \infty$ in Eq. (10.21) and obtain

$$\ln n! + n - \Big(n + \frac{1}{2}\Big) \ln n = \gamma + \frac{1}{12n} - \frac{1}{360n^3} + \widehat{R}_5,$$

where $|\widehat{R}_5| \le 0.00123/n^4$. Taking the exponential function of this expression we get

(10.22) $$n! = D_n \frac{\sqrt{n}\; n^n}{e^n} \qquad \text{with} \qquad D_n = e^{\gamma} \cdot \exp\Big(\frac{1}{12n} - \frac{1}{360n^3} + \widehat{R}_5\Big).$$

This proves (10.18) and also (10.17), as soon as we have seen that the limit of D_n (i.e., $D = e^{\gamma}$) is actually equal to $\sqrt{2\pi}$. To this end, we compute, from (10.22),

$$\frac{D_n \cdot D_n}{D_{2n}} = \frac{n! \cdot n! \cdot (2n)^{2n} \cdot e^{-2n}\sqrt{2n}}{n^{2n} \cdot e^{-2n} \cdot n \cdot (2n)!} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)} \cdot \frac{\sqrt{2}}{\sqrt{n}},$$

which tends to D too. This formula reminds us of Wallis's product of Eq. (I.5.27). Indeed, its square,

$$\Big(\frac{D_n \cdot D_n}{D_{2n}}\Big)^2 = \underbrace{\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ \cdots \ (2n)(2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \ \cdots \ (2n-1)(2n+1)}}_{\to \pi/2} \cdot \underbrace{\frac{2(2n+1)}{n}}_{\to 4},$$

tends to 2π, so that $D = \sqrt{2\pi}$. The stated estimate for $\widetilde{R}_9$ follows from (10.12) and $|B_9(x)| \le 0.04756$. □

The Harmonic Series and Euler's Constant

We try to compute

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

by putting $f(x) = 1/x$ in Theorem 10.2. Since $f^{(j)}(x) = (-1)^j j! x^{-j-1}$, we get, instead of (10.19),

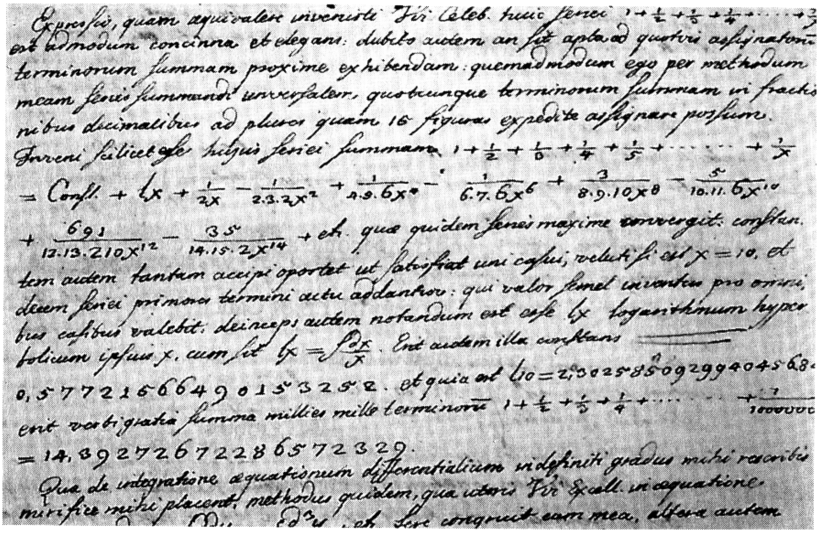
Expressio, quam aequivalentem invenisti Vir Celeb. huius seriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}$
est admodum concinna et elegans: dubito autem an sit apta ad quotvis assignationem
terminorum summam proxime exhibendam: quemadmodum ego per methodum
meam seriei summandi universalem, quotcunque terminorum summam in fracti-
onibus decimalibus ad plures quam 16 figuras expedite assignare possum.
Inveni scilicet esse huius seriei summam $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{x}$
$= \text{Const.} + lx + \frac{1}{2x} - \frac{1}{2.3.2x^2} + \frac{1}{4.5.6x^4} - \frac{1}{6.7.6x^6} + \frac{3}{8.9.10x^8} - \frac{5}{10.11.6x^{10}}$
$+ \frac{691}{12.13.210x^{12}} - \frac{35}{14.15.2x^{14}}$ + etc. quae quidem series maxime convergit. constan-
tem autem tantum accipi oportet ut satisfiat uni casui, veluti si est $x=10$, et
decem seriei priores termini actu addantur: qui valor semel inventus pro omni-
bus casibus valebit. deinceps autem notandum est esse lx logarithmum hyper-
bolicum ipsius x, cum sit $lx = \int \frac{dx}{x}$. Erit autem illa constans
0,57721566490153252. et quia est $l10 = 2,302585092994045684$
erit verbi gratia summa millies mille terminorum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{1000000}$
$= 14,39272672286572329$.
Quae de integratione aequationum differentialium indefiniti gradus mihi rescribis
mirifice mihi placent, methodus quidem, qua uteris Vir Excell. in aequatione

FIGURE 10.2. Euler's autograph (letter to Joh. Bernoulli 1740, see Fellmann 1983, p. 96)[1]

$$\sum_{i=n+1}^{m} \frac{1}{i} = \int_n^m \frac{1}{x}\,dx + \frac{1}{2}\Big(\frac{1}{m} - \frac{1}{n}\Big) - \frac{1}{12}\Big(\frac{1}{m^2} - \frac{1}{n^2}\Big)$$
$$+ \frac{1}{120}\Big(\frac{1}{m^4} - \frac{1}{n^4}\Big) - \frac{1}{252}\Big(\frac{1}{m^6} - \frac{1}{n^6}\Big) + \frac{1}{240}\Big(\frac{1}{m^8} - \frac{1}{n^8}\Big) + \widetilde{R}_9, \tag{10.23}$$

where, because of $|B_9(x)| \le 0.04756$, we have $|\widetilde{R}_9| \le 0.00529/n^9$. The diverging terms to collect will now be, instead of (10.20),

$$\gamma_n = \sum_{i=1}^{n} \frac{1}{i} - \ln n,$$

which is investigated precisely as above and seen to converge. This time, the constant obtained,

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \to \gamma = 0.57721566490153286\ldots, \tag{10.24}$$

is a new constant in mathematics and is called "Euler's constant" (see Fig. 10.2 for an autograph of Euler containing his constant and its use for the computation of the sum of Example 10.1). Letting, as before, $m \to \infty$ in (10.23), we obtain

$$\sum_{i=1}^{n} \frac{1}{i} = \gamma + \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8} + \widetilde{R}_9, \tag{10.25}$$

where $|\widetilde{R}_9| \le 0.00529/n^9$. To find the constant γ, we put, for example, $n = 10$ (as did Euler) in Eq. (10.25) and obtain the value of (10.24). This constant was computed with great precision by D. Knuth (1962). It is still not known whether it is rational or irrational.

Exercises

10.1 The spiral of Theodorus is composed of rectangular triangles of sides 1, $\sqrt{n}$, and $\sqrt{n+1}$. It performs a complete rotation after 17 triangles (this seems to be the reason why Theodorus did not consider roots beyond $\sqrt{17}$). No longer prevented by such scruples, we now want to know how many rotations a billion such triangles perform. This requires the calculation of (see Fig. 10.3)

$$1 + \frac{1}{2\pi} \sum_{i=18}^{1000000000} \arctan \frac{1}{\sqrt{i}}$$

with an error smaller than 1. This exercise is not only a further occasion to admire the power of the Euler-Maclaurin formula, but also leaves us with an interesting integral to evaluate.

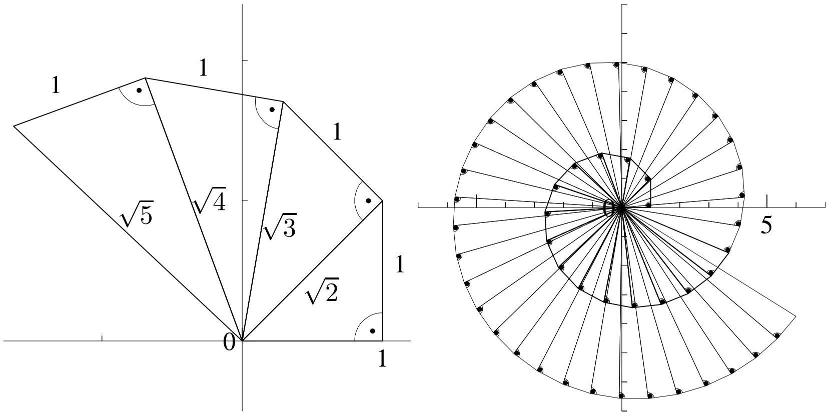

FIGURE 10.3. The spiral of Theodorus of Cyrene, 470–390 B.C.

10.2 (Formula for the Taylor series of $\tan x$). If we let $\cot x = 1/\tan x$ and $\coth x = 1/\tanh x$, Eq. (10.9) can be seen to represent the Taylor series of $(x/2)\coth(x/2)$. This allows us to obtain the series expansion of $x \cdot \coth x$, and, by letting $x \mapsto ix$, that of $x \cdot \cot x$. Finally, use the formula

$$2 \cdot \cot 2x = \cot x - \tan x$$

and obtain the coefficients of the expansion of $\tan x$. Compare it with Eq. (I.4.18).

10.3 Verify numerically the estimates (Lehmer 1940)

$$|B_3(x)| \le 0.04812, \qquad |B_5(x)| \le 0.02446, \qquad |B_7(x)| \le 0.02607,$$
$$|B_9(x)| \le 0.04756, \qquad |B_{11}(x)| \le 0.13250, \qquad |B_{13}(x)| \le 0.52357$$

for $0 \le x \le 1$.