Lecture 21 - An overview of Lie groups

The (unofficial) goal of the last third of the course is to prove no theorems. We'll talk about

1. Lie groups in general,
2. Clifford algebras and Spin groups,
3. Construction of all Lie groups and all representations. You might say this is impossible, so let's just try to do all simple ones, and in particular E_8, E_7, E_6.
4. Representations of $SL_2(\mathbb{R})$.

Lie groups in general

In general, a Lie group G can be broken up into a number of pieces.

The connected component of the identity, $G_{\text{conn}} \subseteq G$, is a normal subgroup, and G/G_{conn} is a discrete group.

$$1 \longrightarrow G_{\text{conn}} \longrightarrow G \longrightarrow G_{\text{discrete}} \longrightarrow 1$$

The maximal connected normal solvable subgroup of G_{conn} is called G_{sol}. Recall that a group is *solvable* if there is a chain of subgroups $G_{\text{sol}} \supseteq \cdots \supseteq 1$, where consecutive quotients are abelian. The Lie algebra of a solvable group is solvable (by Exercise 11.2), so Lie's theorem (Theorem 11.11) tells us that G_{sol} is isomorphic to a subgroup of the group of upper triangular matrices.

Every normal solvable subgroup of $G_{\text{conn}}/G_{\text{sol}}$ is discrete, and therefore in the center (which is itself discrete). We call the pre-image of the center G_*. Then G/G_* is a product of simple groups (groups with no normal subgroups).

$$G_{\text{sol}} \subseteq \left\{ \begin{pmatrix} * & & & \ast \\ & \ddots & & \\ & & * & \\ 0 & & & * \end{pmatrix} \right\} \qquad G_{\text{nil}} \subseteq \left\{ \begin{pmatrix} 1 & & & \ast \\ & \ddots & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \right\}$$

Since G_{sol} is solvable, $G_{\text{nil}} := [G_{\text{sol}}, G_{\text{sol}}]$ is nilpotent, i.e. there is a chain of subgroups $G_{\text{nil}} \supseteq G_1 \supseteq \cdots \supseteq G_k = 1$ such that G_i/G_{i+1} is in the center of G_{nil}/G_{i+1}. In fact, G_{nil} must be isomorphic to a subgroup of the group of upper triangular matrices with ones on the diagonal. Such a group is called *unipotent*.

We have the picture

$$
\begin{array}{rcl}
 & G & \\
 & \big| & \Big\}\,\text{discrete; classification hopeless} \\
\left\{\right. & G_{\text{conn}} & \\
 & \big| & \Big\}\,\prod\text{connected simples; classified} \\
 & G_* & \\
 & \big| & \Big\}\,\text{abelian discrete} \\
\text{connected} & G_{\text{sol}} & \qquad\qquad\qquad\quad \Big\}\,\text{classification trivial} \\
 & \big| & \Big\}\,\text{abelian} \\
 & G_{\text{nil}} & \\
 & \big| & \Big\}\,\text{nilpotent; classification a mess} \\
\left.\right\} & 1 &
\end{array}
$$

The classification of connected simple Lie groups is quite long. There are many infinite series and a lot of exceptional cases. Some infinite series are $PSU(n)$, $PSL_n(\mathbb{R})$, and $PSL_n(\mathbb{C})$.[1]

One way to get many connected simple Lie groups is not observe that there is a unique connected simple Lie group for each simple Lie algebra. We've already classified complex Lie algebras, and it turns out that there a finite number of real Lie algebras which complexify to any given complex Lie algebra. We will classify all such real forms in Lecture 29.

For example, $\mathfrak{sl}_2(\mathbb{R}) \not\simeq \mathfrak{su}_2(\mathbb{R})$, but $\mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} \simeq \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$. By the way, $\mathfrak{sl}_2(\mathbb{C})$ is simple as a *real* Lie algebra, but its complexification is $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, which is not simple. Thus, we cannot obtain all connected simple groups this way.

Example 21.1. Let G be the group of all shape-preserving transformations of $\mathbb{R}^4$ (i.e. translations, reflections, rotations, and scaling). It is sometimes called $\mathbb{R}^4 \cdot GO_4(\mathbb{R})$. The $\mathbb{R}^4$ stands for translations, the G means that you can multiply by scalars, and the O means that you can reflect and rotate. The $\mathbb{R}^4$ is a normal subgroup. In this case, we have

$$
\begin{array}{rrl}
 & \mathbb{R}^4 \cdot GO_4(\mathbb{R}) = G & \\
 & & G/G_{\text{conn}} = \mathbb{Z}/2\mathbb{Z} \\
 & \mathbb{R}^4 \cdot GO_4^+(\mathbb{R}) = G_{\text{conn}} & \\
 & & G_{\text{conn}}/G_* = PSO_4(\mathbb{R}) \\
\begin{array}{l} G_{\text{conn}}/G_{\text{sol}} \\ = SO_4(\mathbb{R}) \end{array} & \mathbb{R}^4 \cdot \mathbb{R}^\times = G_* & \qquad\qquad \big(\simeq SO_3(\mathbb{R}) \times SO_3(\mathbb{R})\big) \\
 & & G_*/G_{\text{sol}} = \mathbb{Z}/2\mathbb{Z} \\
 & \mathbb{R}^4 \cdot \mathbb{R}^+ = G_{\text{sol}} & \\
 & & G_{\text{sol}}/G_{\text{nil}} = \mathbb{R}^+ \\
 & \mathbb{R}^4 = G_{\text{nil}} &
\end{array}
$$

[1] The P means "mod out by the center".

where $GO_4^+(\mathbb{R})$ is the connected component of the identity (those transformations that preserve orientation), $\mathbb{R}^\times$ is scaling by something other than zero, and $\mathbb{R}^+$ is scaling by something positive. Note that $SO_3(\mathbb{R}) = PSO_3(\mathbb{R})$ is simple.

$SO_4(\mathbb{R})$ is "almost" the product $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. To see this, consider the associative (but not commutative) algebra of quaternions, $\mathbb{H}$. Since $q\bar{q} = a^2 + b^2 + c^2 + d^2 > 0$ whenever $q \neq 0$, any non-zero quaternion has an inverse (namely, $\bar{q}/q\bar{q}$). Thus, $\mathbb{H}$ is a division algebra. Think of $\mathbb{H}$ as $\mathbb{R}^4$ and let S^3 be the unit sphere, consisting of the quaternions such that $\|q\| = q\bar{q} = 1$. It is easy to check that $\|pq\| = \|p\| \cdot \|q\|$, from which we get that left (right) multiplication by an element of S^3 is a norm-preserving transformation of $\mathbb{R}^4$. So we have a map $S^3 \times S^3 \to O_4(\mathbb{R})$. Since $S^3 \times S^3$ is connected, the image must lie in $SO_4(\mathbb{R})$. It is not hard to check that $SO_4(\mathbb{R})$ is the image. The kernel is $\{(1,1), (-1,-1)\}$. So we have $S^3 \times S^3/\{(1,1), (-1,-1)\} \simeq SO_4(\mathbb{R})$.

Conjugating a purely imaginary quaternion by some $q \in S^3$ yields a purely imaginary quaternion of the same norm as the original, so we have a homomorphism $S^3 \to O_3(\mathbb{R})$. Again, it is easy to check that the image is $SO_3(\mathbb{R})$ and that the kernel is ± 1, so $S^3/\{\pm 1\} \simeq SO_3(\mathbb{R})$.

So the universal cover of $SO_4(\mathbb{R})$ (a double cover) is the cartesian square of the universal cover of $SO_3(\mathbb{R})$ (also a double cover). Orthogonal groups in dimension 4 have a strong tendency to split up like this. Orthogonal groups in general tend to have these double covers, as we shall see in Lectures 23 and 24. These double covers are important if you want to study fermions.

Lie groups and Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. We can set $\mathfrak{g}_{\text{sol}} = \operatorname{rad} \mathfrak{g}$ to be the maximal solvable ideal (normal subalgebra), and $\mathfrak{g}_{\text{nil}} = [\mathfrak{g}_{\text{sol}}, \mathfrak{g}_{\text{sol}}]$. Then we get the chain

$$\begin{array}{ll}
\mathfrak{g} & \\
\Big| & \Big\} \prod \text{simples; classification known} \\
\mathfrak{g}_{\text{sol}} & \\
\Big| & \Big\} \text{abelian; easy to classify} \\
\mathfrak{g}_{\text{nil}} & \\
\Big| & \Big\} \text{nilpotent; classification a mess} \\
0 &
\end{array}$$

We have an equivalence of categories between simply connected Lie groups and Lie algebras. The correspondence cannot detect

– Non-trivial components of G. For example, SO_n and O_n have the same Lie algebra.

– Discrete normal (therefore central, Lemma 5.1) subgroups of G. If $Z \subseteq G$ is any discrete normal subgroup, then G and G/Z have the same Lie algebra. For example, $SU(2)$ has the same Lie algebra as $PSU(2) \simeq SO_3(\mathbb{R})$.

If $\tilde{G}$ is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, then any other connected group G with Lie algebra $\mathfrak{g}$ must be isomorphic to $\tilde{G}/Z$, where Z is some discrete subgroup of the center. Thus, if you know all the discrete subgroups of the center of $\tilde{G}$, you can read off all the connected Lie groups with the given Lie algebra.

Let's find all the groups with the algebra $\mathfrak{so}_4(\mathbb{R})$. First let's find a simply connected group with this Lie algebra. You might guess $SO_4(\mathbb{R})$, but that isn't simply connected. The simply connected one is $S^3 \times S^3$ as we saw earlier (it is a product of two simply connected groups, so it is simply connected). The center of S^3 is generated by -1, so the center of $S^3 \times S^3$ is $(\mathbb{Z}/2\mathbb{Z})^2$, the Klein four group. There are three subgroups of order 2

$$
\begin{tikzcd}
 & (\mathbb{Z}/2\mathbb{Z})^2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(-1,1) \arrow[dr, dash] & (-1,-1) \arrow[d, dash] & (1,-1) \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & PSO_4(\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
SO_3(\mathbb{R}) \times S^3 \arrow[dr, dash] & SO_4(\mathbb{R}) \arrow[d, dash] & S^3 \times SO_3(\mathbb{R}) \arrow[dl, dash] \\
 & S^3 \times S^3 &
\end{tikzcd}
$$

Therefore, there are 5 groups with Lie algebra $\mathfrak{so}_4$.

Lie groups and finite groups

1. The classification of finite simple groups resembles the classification of connected simple Lie groups when $n \geq 2$.

   For example, $PSL_n(\mathbb{R})$ is a simple Lie group, and $PSL_n(\mathbb{F}_q)$ is a finite simple group except when $n = q = 2$ or $n = 2, q = 3$. Simple finite groups form about 18 series similar to Lie groups, and 26 or 27 exceptions, called sporadic groups, which don't seem to have any analogues for Lie groups.

2. Finite groups and Lie groups are both built up from simple and abelian groups. However, the way that finite groups are built is much more complicated than the way Lie groups are built. Finite groups can contain simple subgroups in very complicated ways; not just as direct factors.

   For example, there are *wreath products*. Let G and H be finite simple groups with an action of H on a set of n points. Then H

acts on G^n by permuting the factors. We can form the semi-direct product $G^n \ltimes H$, sometimes denoted $G \wr H$. There is no analogue for (finite dimensional) Lie groups. There *is* an analogue for infinite dimensional Lie groups, which is why the theory becomes hard in infinite dimensions.

3. The commutator subgroup of a solvable finite group need not be a nilpotent group. For example, the symmetric group S_4 has commutator subgroup A_4, which is not nilpotent.

Lie groups and Algebraic groups (over $\mathbb{R}$)

By algebraic group, we mean an algebraic variety which is also a group, such as $GL_n(\mathbb{R})$. Any algebraic group is a Lie group. Probably all the Lie groups you've come across have been algebraic groups. Since they are so similar, we'll list some differences.

1. Unipotent and semisimple abelian algebraic groups are totally different, but for Lie groups they are nearly the same. For example $\mathbb{R} \simeq \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ is unipotent and $\mathbb{R}^\times \simeq \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right\}$ is semisimple. As Lie groups, they are closely related (nearly the same), but the Lie group homomorphism $\exp : \mathbb{R} \to \mathbb{R}^\times$ is not algebraic (polynomial), so they look quite different as algebraic groups.

2. Abelian varieties are different from affine algebraic groups. For example, consider the (projective) elliptic curve $y^2 = x^3 + x$ with its usual group operation and the group of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $a^2 + b^2 = 1$. Both are isomorphic to S^1 as Lie groups, but they are completely different as algebraic groups; one is projective and the other is affine.

3. Some Lie groups do not correspond to ANY algebraic group. We give two examples here.

   The *Heisenberg group* is the subgroup of symmetries of $L^2(\mathbb{R})$ generated by translations ($f(t) \mapsto f(t + x)$), multiplication by $e^{2\pi i t y}$ ($f(t) \mapsto e^{2\pi i t y} f(t)$), and multiplication by $e^{2\pi i z}$ ($f(t) \mapsto e^{2\pi i z} f(t)$). The general element is of the form $f(t) \mapsto e^{2\pi i(yt+z)} f(t + x)$. This can also be modelled as

$$\left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right\} \Big/ \left\{\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z}\right\}$$

   It has the property that in any finite dimensional representation, the center (elements with $x = y = 0$) acts trivially, so it cannot be isomorphic to any algebraic group.

The *metaplectic group*. Let's try to find all connected groups with Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a + d = 0\}$. There are two obvious ones: $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$. There aren't any other ones that can be represented as groups of finite dimensional matrices. However, if you look at $SL_2(\mathbb{R})$, you'll find that it is not simply connected. To see this, we will use Iwasawa decomposition (without proof).

Theorem 21.2 (Iwasawa decomposition)**.** *If G is a (connected) semisimple Lie group, then there are closed subgroups K, A, and N, with K compact, A abelian, and N unipotent, such that the multiplication map $K \times A \times N \to G$ is a surjective diffeomorphism. Moreover, A and N are simply connected.*

In the case of SL_n, this is the statement that any basis can be obtained uniquely by taking an orthonormal basis ($K = SO_n$), scaling by positive reals (A is the group of diagonal matrices with positive real entries), and shearing (N is the group $\left(\begin{smallmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}\right)$). This is exactly the result of the Gram-Schmidt process.

The upshot is that $G \simeq K \times A \times N$ (topologically), and A and N do not contribute to the fundamental group, so the fundamental group of G is the same as that of K. In our case, $K = SO_2(\mathbb{R})$ is isomorphic to a circle, so the fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$.

So the universal cover $\widetilde{SL_2(\mathbb{R})}$ has center $\mathbb{Z}$. Any finite dimensional representation of $\widetilde{SL_2}(\mathbb{R})$ factors through $SL_2(\mathbb{R})$, so none of the covers of $SL_2(\mathbb{R})$ can be written as a group of finite dimensional matrices. Representing such groups is a pain.

The most important case is the metaplectic group $Mp_2(\mathbb{R})$, which is the connected double cover of $SL_2(\mathbb{R})$. It turns up in the theory of modular forms of half-integral weight and has a representation called the metaplectic representation.

Important Lie groups

<u>Dimension 1</u>: There are just $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$.

<u>Dimension 2</u>: The abelian groups are quotients of $\mathbb{R}^2$ by some discrete subgroup; there are three cases: $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z} = \mathbb{R} \times S^1$, and $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$.

There is also a non-abelian group, the group of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$, where $a > 0$. The Lie algebra is the subalgebra of 2×2 matrices of the form $\left(\begin{smallmatrix} h & x \\ 0 & -h \end{smallmatrix}\right)$, which is generated by two elements H and X, with $[H, X] = 2X$.

Dimension 3: There are some boring abelian and solvable groups, such as $\mathbb{R}^2 \ltimes \mathbb{R}^1$, or the direct sum of $\mathbb{R}^1$ with one of the two dimensional groups. As the dimension increases, the number of boring solvable groups gets huge, and nobody can do anything about them, so we ignore them from here on.

You get the group $SL_2(\mathbb{R})$, which is the most important Lie group of all. We saw earlier that $SL_2(\mathbb{R})$ has fundamental group $\mathbb{Z}$. The double cover $Mp_2(\mathbb{R})$ is important. The quotient $PSL_2(\mathbb{R})$ is simple, and acts on the open upper half plane by linear fractional transformations

Closely related to $SL_2(\mathbb{R})$ is the compact group SU_2. We know that $SU_2 \simeq S^3$, and it covers $SO_3(\mathbb{R})$, with kernel ± 1. After we learn about Spin groups, we will see that $SU_2 \cong \mathrm{Spin}_3(\mathbb{R})$. The Lie algebra $\mathfrak{su}_2$ is generated by three elements X, Y, and Z with relations $[X,Y]=2Z$, $[Y,Z]=2X$, and $[Z,X]=2Y$.[2]

The Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ are non-isomorphic, but when you complexify, they both become isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

There is another interesting 3 dimensional algebra. The Heisenberg algebra is the Lie algebra of the Heisenberg group. It is generated by X, Y, Z, with $[X,Y]=Z$ and Z central. You can think of this as strictly upper triangular matrices.

Dimension 6: (nothing interesting happens in dimensions 4,5) We get the group $SL_2(\mathbb{C})$. Later, we will see that it is also called $\mathrm{Spin}_{1,3}(\mathbb{R})$.

Dimension 8: We have $SU_3(\mathbb{R})$ and $SL_3(\mathbb{R})$. This is the first time we get a non-trivial root system.

Dimension 14: G_2, which we will discuss a little.

Dimension 248: E_8, which we will discuss in detail.

This class is mostly about finite dimensional algebras, but let's mention some infinite dimensional Lie groups or Lie algebras.

1. Automorphisms of a Hilbert space form a Lie group.
2. Diffeomorphisms of a manifold form a Lie group. There is some physics stuff related to this.
3. *Gauge groups* are (continuous, smooth, analytic, or whatever) maps from a manifold M to a group G.
4. The *Virasoro algebra* is generated by L_n for $n \in \mathbb{Z}$ and c, with relations $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}c$, where c is central (called the *central charge*). If you set $c=0$, you get (complexified) vector fields on S^1, where we think of L_n as $ie^{in\theta}\frac{\partial}{\partial\theta}$. Thus, the

[2] An explicit representation is given by $X = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, and $Z = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. The cross product on $\mathbb{R}^3$ gives it the structure of this Lie algebra.

Virasoro algebra is a central extension

$$0 \to c\mathbb{C} \to \text{Virasoro} \to \text{Vect}(S^1) \to 0.$$

5. Affine Kac-Moody algebras, which are more or less central extensions of certain gauge groups over the circle.

Lecture 22 - Clifford algebras

With Lie algebras of small dimensions, there are accidental isomorphisms. Almost all of these can be explained with Clifford algebras and Spin groups.

Motivational examples that we'd like to explain:

1. $SO_2(\mathbb{R}) = S^1$: S^1 can double cover S^1 itself.

2. $SO_3(\mathbb{R})$: has a simply connected double cover S^3.

3. $SO_4(\mathbb{R})$: has a simply connected double cover $S^3 \times S^3$.

4. $SO_5(\mathbb{C})$: Look at $Sp_4(\mathbb{C})$, which acts on $\mathbb{C}^4$ and on $\Lambda^2(\mathbb{C}^4)$, which is 6 dimensional, and decomposes as $5 \oplus 1$. $\Lambda^2(\mathbb{C}^4)$ has a symmetric bilinear form given by $\Lambda^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{C}^4) \to \Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$, and $Sp_4(\mathbb{C})$ preserves this form. You get that $Sp_4(\mathbb{C})$ acts on $\mathbb{C}^5$, preserving a symmetric bilinear form, so it maps to $SO_5(\mathbb{C})$. You can check that the kernel is ± 1. So $Sp_4(\mathbb{C})$ is a double cover of $SO_5(\mathbb{C})$.

5. $SO_5(\mathbb{C})$: $SL_4(\mathbb{C})$ acts on $\mathbb{C}^4$, and we still have our 6 dimensional $\Lambda^2(\mathbb{C}^4)$, with a symmetric bilinear form. So you get a homomorphism $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$, which you can check is surjective, with kernel ± 1.

So we have double covers S^1, S^3, $S^3 \times S^3$, $Sp_4(\mathbb{C})$, $SL_4(\mathbb{C})$ of the orthogonal groups in dimensions 2,3,4,5, and 6, respectively. All of these look completely unrelated. By the end of the next lecture, we will have an understanding of these groups, which will be called $\mathrm{Spin}_2(\mathbb{R})$, $\mathrm{Spin}_3(\mathbb{R})$, $\mathrm{Spin}_4(\mathbb{R})$, $\mathrm{Spin}_5(\mathbb{C})$, and $\mathrm{Spin}_6(\mathbb{C})$, respectively.

Example 22.1. We have not yet defined Clifford algebras, but here are some examples of Clifford algebras over $\mathbb{R}$.

- $\mathbb{C}$ is generated by $\mathbb{R}$, together with i, with $i^2 = -1$

- $\mathbb{H}$ is generated by $\mathbb{R}$, together with i, j, each squaring to -1, with $ij + ji = 0$.

- Dirac wanted a square root for the operator $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$ (the wave operator in 4 dimensions). He supposed that the square root is of the form $A = \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + \gamma_4 \frac{\partial}{\partial t}$ and compared coefficients in the equation $A^2 = \nabla$. Doing this yields $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1$, $\gamma_4^2 = -1$, and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$ for $i \neq j$.

  Dirac solved this by taking the γ_i to be 4×4 complex matrices. A operates on vector-valued functions on space-time.

Definition 22.2. A general *Clifford algebra* over $\mathbb{R}$ should be generated by elements $\gamma_1, \ldots, \gamma_n$ such that γ_i^2 is some given real, and $\gamma_i\gamma_j + \gamma_j\gamma_i = 0$ for $i \neq j$.

Definition 22.3 (better definition)**.** Suppose V is a vector space over a field K, with some quadratic form[1] $N : V \to K$. Then the *Clifford algebra* $C_V(K)$ is generated by the vector space V, with relations $v^2 = N(v)$.

We know that $N(\lambda v) = \lambda^2 N(v)$ and that the expression $(a, b) := N(a+b) - N(a) - N(b)$ is bilinear. If the characteristic of K is not 2, we have $N(a) = \frac{(a,a)}{2}$. Thus, you can work with symmetric bilinear forms instead of quadratic forms so long as the characteristic of K is not 2. We'll use quadratic forms so that everything works in characteristic 2.

Warning 22.4. A few authors (mainly in index theory) use the relations $v^2 = -N(v)$. Some people add a factor of 2, which usually doesn't matter, but is wrong in characteristic 2.

Example 22.5. Take $V = \mathbb{R}^2$ with basis i, j, and with $N(xi + yj) = -x^2 - y^2$. Then the relations are $(xi + yj)^2 = -x^2 - y^2$ are exactly the relations for the quaternions: $i^2 = j^2 = -1$ and $(i+j)^2 = i^2+ij+ji+j^2 = -2$, so $ij + ji = 0$.

Remark 22.6. If the characteristic of K is not 2, a "completing the square" argument shows that any quadratic form is isomorphic to $c_1x_1^2 + \cdots + c_nx_n^2$, and if one can be obtained from another other by permuting the c_i and multiplying each c_i by a non-zero square, the two forms are isomorphic.

It follows that every quadratic form on a vector space over $\mathbb{C}$ is isomorphic to $x_1^2 + \cdots + x_n^2$, and that every quadratic form on a vector space over $\mathbb{R}$ is isomorphic to $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$ (m pluses and n minuses) for some m and n. One can check that these forms over $\mathbb{R}$ are non-isomorphic.

We will always assume that N is non-degenerate (i.e. that the associated bilinear form is non-degenerate), but one could study Clifford algebras arising from degenerate forms.

Warning 22.7. The criterion in the remark is not sufficient for classifying quadratic forms. For example, over the field $\mathbb{F}_3$, the forms x^2+y^2 and $-x^2-y^2$ are isomorphic via the isomorphism $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) : \mathbb{F}_3^2 \to \mathbb{F}_3^2$, but -1 is not a square in $\mathbb{F}_3$. Also, completing the square doesn't work in characteristic 2.

[1] N is a *quadratic form* if it is a homogeneous polynomial of degree 2 in the coefficients with respect to some basis.

Remark 22.8. The tensor algebra TV has a natural $\mathbb{Z}$-grading, and to form the Clifford algebra $C_V(K)$, we quotient by the ideal generated by the even elements $v^2 - N(v)$. Thus, the algebra $C_V(K) = C_V^0(K) \oplus C_V^1(K)$ is $\mathbb{Z}/2\mathbb{Z}$-graded. A $\mathbb{Z}/2\mathbb{Z}$-graded algebra is called a *superalgebra*.

Problem: Find the structure of $C_{m,n}(\mathbb{R})$, the Clifford algebra over $\mathbb{R}^{n+m}$ with the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$.

Example 22.9.

- $C_{0,0}(\mathbb{R})$ is $\mathbb{R}$.
- $C_{1,0}(\mathbb{R})$ is $\mathbb{R}[\varepsilon]/(\varepsilon^2 - 1) = \mathbb{R}(1+\varepsilon) \oplus \mathbb{R}(1-\varepsilon) = \mathbb{R} \oplus \mathbb{R}$. Note that the given basis, this is a direct sum of *algebras* over $\mathbb{R}$.
- $C_{0,1}(\mathbb{R})$ is $\mathbb{R}[i]/(i^2+1) = \mathbb{C}$, with i odd.
- $C_{2,0}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 - 1, \alpha\beta + \beta\alpha)$. We get a homomorphism $C_{2,0}(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The homomorphism is onto because the two given matrices generate $\mathbb{M}_2(\mathbb{R})$ as an algebra. The dimension of $\mathbb{M}_2(\mathbb{R})$ is 4, and the dimension of $C_{2,0}(\mathbb{R})$ is at most 4 because it is spanned by 1, α, β, and $\alpha\beta$. So we have that $C_{2,0}(\mathbb{R}) \simeq \mathbb{M}_2(\mathbb{R})$.
- $C_{1,1}(\mathbb{R})$ is $\mathbb{R}[\alpha, \beta]/(\alpha^2 - 1, \beta^2 + 1, \alpha\beta + \beta\alpha)$. Again, we get an isomorphism with $\mathbb{M}_2(\mathbb{R})$, given by $\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\beta \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

Thus, we've computed the Clifford algebras

$m\backslash n$	0	1	2
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{M}_2(\mathbb{R})$	
2	$\mathbb{M}_2(\mathbb{R})$		

Remark 22.10. If $\{v_1, \dots, v_n\}$ is a basis for V, then $\{v_{i_1} \cdots v_{i_k} | i_1 < \cdots < i_k,\ k \le n\}$ spans $C_V(K)$, so the dimension of $C_V(K)$ is less than or equal to $2^{\dim V}$. The tough part of Clifford algebras is showing that it cannot be smaller.

Now let's try to analyze larger Clifford algebras more systematically. What is $C_{U\oplus V}$ in terms of C_U and C_V? One might guess $C_{U\oplus V} \cong C_U \otimes C_V$. For the usual definition of tensor product, this is false (e.g. $C_{1,1}(\mathbb{R}) \neq C_{1,0}(\mathbb{R}) \otimes C_{0,1}(\mathbb{R})$). However, for the *superalgebra* definition of tensor product, this is correct. The superalgebra tensor product is the regular tensor product of vector spaces, with the product given by $(a \otimes b)(c \otimes d) = (-1)^{\deg b \cdot \deg c} ac \otimes bd$ for homogeneous elements a, b, c, and d.

Let's specialize to the case $K = \mathbb{R}$ and try to compute $C_{U\oplus V}(K)$. Assume for the moment that $\dim U = m$ is even. Take $\alpha_1, \ldots, \alpha_m$ to be an orthogonal basis for U and let $\beta_1, \ldots, \beta_n$ to be an orthogonal basis for V. Then set $\gamma_i = \alpha_1\alpha_2\cdots\alpha_m\beta_i$. What are the relations between the α_i and the γ_j? We have

$$\alpha_i\gamma_j = \alpha_i\alpha_1\alpha_2\cdots\alpha_m\beta_j = \alpha_1\alpha_2\cdots\alpha_m\beta_i\alpha_i = \gamma_j\alpha_i$$

since $\dim U$ is even, and α_i anti-commutes with everything except itself.

$$\begin{aligned}
\gamma_i\gamma_j &= \gamma_i\alpha_1\cdots\alpha_m\beta_j = \alpha_1\cdots\alpha_m\gamma_i\beta_j \\
&= \alpha_1\cdots\alpha_m\alpha_1\cdots\alpha_m \underbrace{\beta_i\beta_j}_{-\beta_j\beta_i} = -\gamma_j\gamma_i \\
\gamma_i^2 &= \alpha_1\cdots\alpha_m\alpha_1\cdots\alpha_m\beta_i\beta_i = (-1)^{\frac{m(m-1)}{2}}\alpha_1^2\cdots\alpha_m^2\beta_i^2 \\
&= (-1)^{m/2}\alpha_1^2\cdots\alpha_m^2\beta_i^2 \qquad\qquad (m \text{ even})
\end{aligned}$$

So the γ_i's commute with the α_i and satisfy the relations of some Clifford algebra. Thus, we've shown that $C_{U\oplus V}(K) \cong C_U(K) \otimes C_W(K)$, where W is V with the quadratic form multiplied by $(-1)^{\frac{1}{2}\dim U}\alpha_1^2\cdots\alpha_m^2 = (-1)^{\frac{1}{2}\dim U}\cdot$ discriminant(U), and this is the usual tensor product of algebras over $\mathbb{R}$.

Taking $\dim U = 2$, we find that

$$\begin{aligned}
C_{m+2,n}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{n,m}(\mathbb{R}) \\
C_{m+1,n+1}(\mathbb{R}) &\cong \mathbb{M}_2(\mathbb{R}) \otimes C_{m,n}(\mathbb{R}) \\
C_{m,n+2}(\mathbb{R}) &\cong \mathbb{H} \otimes C_{n,m}(\mathbb{R})
\end{aligned}$$

where the indices switch whenever the discriminant is positive. Using these formulas, we can reduce any Clifford algebra to tensor products of things like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{M}_2(\mathbb{R})$.

Recall the rules for taking tensor products of matrix algebras (all tensor products are over $\mathbb{R}$).

– $\mathbb{R} \otimes X \cong X$.

– $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{M}_2(\mathbb{C})$.

  This follows from the isomorphism $\mathbb{C} \otimes C_{m,n}(\mathbb{R}) \cong C_{m+n}(\mathbb{C})$.

– $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

– $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{M}_4(\mathbb{R})$.

  You can see by thinking of the action on $\mathbb{H} \cong \mathbb{R}^4$ given by $(x \otimes y) \cdot z = xzy^{-1}$.

– $\mathbb{M}_m\big(\mathbb{M}_n(X)\big) \cong \mathbb{M}_{mn}(X)$.

– $\mathbb{M}_m(X) \otimes \mathbb{M}_n(Y) \cong \mathbb{M}_{mn}(X \otimes Y)$.

Filling in the middle of the table is easy because you can move diagonally by tensoring with $\mathbb{M}_2(\mathbb{R})$. It is easy to see that $C_{8+m,n}(\mathbb{R}) \cong C_{m,n+8}(\mathbb{R}) \cong C_{m,n} \otimes \mathbb{M}_{16}(\mathbb{R})$, which gives the table a kind of mod 8 periodicity. There is a more precise way to state this: $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are *super Morita equivalent* if and only if $m - n \equiv m' - n' \mod 8$.

$$
\begin{tikzcd}[column sep=small]
m \backslash n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & \mathbb{R} \arrow[rrrrrrrr, bend left, "\otimes \mathbb{M}_{16}"] \arrow[dddddddd, bend right, "\otimes \mathbb{M}_{16}"'] & \mathbb{C} \arrow[dddl, bend left, "\otimes \mathbb{M}_2"] & \mathbb{H} \arrow[ddddll, bend left] & \mathbb{H} \oplus \mathbb{H} & \mathbb{M}_2(\mathbb{H}) & \mathbb{M}_4(\mathbb{C}) & \mathbb{M}_8(\mathbb{R}) & \mathbb{M}_8(\mathbb{R}) \oplus \mathbb{M}_8(\mathbb{R}) & \mathbb{M}_{16}(\mathbb{R}) \\
1 & \mathbb{R} \oplus \mathbb{R} \arrow[urrr, bend right, "\otimes \mathbb{H}"'] & & & & & & & & \\
2 & \mathbb{M}_2(\mathbb{R}) \arrow[uurrrr, bend right] & & & & & & & & \\
3 & \mathbb{M}_2(\mathbb{C}) & & & & & & & & \\
4 & \mathbb{M}_2(\mathbb{H}) & & & & & & & & \\
5 & \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_2(\mathbb{H}) \arrow[dr, "\otimes \mathbb{M}_2"] & & & & & & & & \\
6 & \mathbb{M}_4(\mathbb{H}) & \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{H}) \arrow[dr, "\otimes \mathbb{M}_2"] & & & & & & & \\
7 & \mathbb{M}_8(\mathbb{C}) & & \text{etc.} & & & & & & \\
8 & \mathbb{M}_{16}(\mathbb{R}) & & & & & & & & \\
9 & \mathbb{M}_{16}(\mathbb{R}) \oplus \mathbb{M}_{16}(\mathbb{R}) & & & & & & & &
\end{tikzcd}
$$

Lecture 23

Last time we defined the Clifford algebra $C_V(K)$, where V is a vector space over K with a quadratic form N. $C_V(K)$ is generated by V with $x^2 = N(x)$. $C_{m,n}(\mathbb{R})$ uses the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$. We found that the structure depends heavily on $m - n \mod 8$.

Remark 23.1. This mod 8 periodicity turns up in several other places:

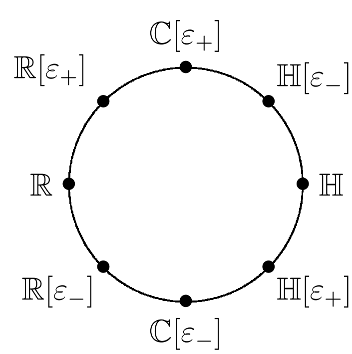
1. Real Clifford algebras $C_{m,n}(\mathbb{R})$ and $C_{m',n'}(\mathbb{R})$ are super Morita equivalent if and only if $m - n \equiv m' - n' \mod 8$.

2. *Bott periodicity*, which says that stable homotopy groups of orthogonal groups are periodic mod 8.

3. Real K-theory is periodic with a period of 8.

4. Even unimodular lattices (such as the E_8 lattice) exist in $\mathbb{R}^{m,n}$ if and only if $m - n \equiv 0 \mod 8$.

5. The Super Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8\mathbb{Z}$. The Super Brauer group consists of super division algebras over $\mathbb{R}$ (algebras in which every non-zero homogeneous element is invertible) with the operation of tensor product modulo super Morita equivalence.[1]



where $\varepsilon_\pm$ are odd with $\varepsilon_\pm^2 = \pm 1$, and $i \in \mathbb{C}$ is odd,[2] but $i, j, k \in \mathbb{H}$ are even.

Recall that $C_V(\mathbb{R}) = C_V^0(\mathbb{R}) \oplus C_V^1(\mathbb{R})$, where $C_V^1(\mathbb{R})$ is the odd part and $C_V^0(\mathbb{R})$ is the even part. It turns out that we will need to know the structure of $C_{m,n}^0(\mathbb{R})$. Fortunately, this is easy to compute in terms of smaller Clifford algebras. Let $\dim U = 1$, with γ a basis for U and let

[1] See http://math.ucr.edu/home/baez/trimble/superdivision.html

[2] One could make i even since $\mathbb{R}[i, \varepsilon_\pm] = \mathbb{R}[\mp\varepsilon_\pm i, \varepsilon_\pm]$, and $\mathbb{R}[\mp\varepsilon_\pm i] \cong \mathbb{C}$ is entirely even.

$\gamma_1, \ldots, \gamma_n$ an orthogonal basis for V. Then $C^0_{U\oplus V}(K)$ is generated by $\gamma\gamma_1, \ldots, \gamma\gamma_n$. We compute the relations

$$\gamma\gamma_i \cdot \gamma\gamma_j = -\gamma\gamma_j \cdot \gamma\gamma_i$$

for $i \neq j$, and

$$(\gamma\gamma_i)^2 = (-\gamma^2)\gamma_i^2$$

So $C^0_{U\oplus V}(K)$ is itself the Clifford algebra $C_W(K)$, where W is V with the quadratic form multiplied by $-\gamma^2 = -\mathrm{disc}(U)$. Over $\mathbb{R}$, this tells us that

$$\begin{aligned} C^0_{m+1,n}(\mathbb{R}) &\cong C_{n,m}(\mathbb{R}) \qquad \text{(mind the indices)} \\ C^0_{m,n+1}(\mathbb{R}) &\cong C_{m,n}(\mathbb{R}). \end{aligned}$$

Remark 23.2. For complex Clifford algebras, the situation is similar, but easier. One finds that $C_{2m}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C})$ and $C_{2m+1}(\mathbb{C}) \cong \mathbb{M}_{2^m}(\mathbb{C}) \oplus \mathbb{M}_{2^m}(\mathbb{C})$, with $C^0_n(\mathbb{C}) \cong C_{n-1}(\mathbb{C})$. You could figure these out by tensoring the real algebras with $\mathbb{C}$ if you wanted. We see a mod 2 periodicity now. Bott periodicity for the unitary group is mod 2.

Clifford groups, Spin groups, and Pin groups

In this section, we define Clifford groups, denoted $\Gamma_V(K)$, and find an exact sequence

$$1 \to K^\times \xrightarrow{\text{central}} \Gamma_V(K) \to O_V(K) \to 1.$$

Definition 23.3. $\Gamma_V(K) = \{x \in C_V(K)$ homogeneous[3]$|xV\alpha(x)^{-1} \subseteq V\}$ (recall that $V \subseteq C_V(K)$), where α is the automorphism of $C_V(K)$ induced by -1 on V (i.e. the automorphism which acts by -1 on odd elements and 1 on even elements).

Note that $\Gamma_V(K)$ acts on V by $x \cdot v = xv\alpha(x)^{-1}$.

Many books leave out the α, which is a mistake, though not a serious one. They use xVx^{-1} instead of $xV\alpha(x)^{-1}$. Our definition is better for the following reasons:

1. It is the correct superalgebra sign. The superalgebra convention says that whenever you exchange two elements of odd degree, you pick up a minus sign, and V is odd.

[3]We assume that $\Gamma_V(K)$ consists of homogeneous elements, but this can actually be proven.

2. Putting α in makes the theory much cleaner in odd dimensions. For example, we will see that the described action gives a map $\Gamma_V(K) \to O_V(K)$ which is onto if we use α, but not if we do not. (You get $SO_V(K)$ without the α, which isn't too bad, but is still annoying.)

Lemma 23.4.[4] *The elements of $\Gamma_V(K)$ which act trivially on V are the elements of $K^\times \subseteq \Gamma_V(K) \subseteq C_V(K)$.*

Proof. Suppose $a_0 + a_1 \in \Gamma_V(K)$ acts trivially on V, with a_0 even and a_1 odd. Then $(a_0 + a_1)v = v\alpha(a_0 + a_1) = v(a_0 - a_1)$. Matching up even and odd parts, we get $a_0 v = v a_0$ and $a_1 v = -v a_1$. Choose an orthogonal basis $\gamma_1, \ldots, \gamma_n$ for V.[5] We may write

$$a_0 = x + \gamma_1 y$$

where $x \in C_V^0(K)$ and $y \in C_V^1(K)$ and neither x nor y contain a factor of γ_1, so $\gamma_1 x = x\gamma_1$ and $\gamma_1 y = y\gamma_1$. Applying the relation $a_0 v = v a_0$ with $v = \gamma_1$, we see that $y = 0$, so a_0 contains no monomials with a factor γ_1.

Repeat this procedure with v equal to the other basis elements to show that $a_0 \in K^\times$ (since it cannot have any γ's in it). Similarly, write $a_1 = y + \gamma_1 x$, with x and y not containing a factor of γ_1. Then the relation $a_1\gamma_1 = -\gamma_1 a_1$ implies that $x = 0$. Repeating with the other basis vectors, we conclude that $a_1 = 0$.

So $a_0 + a_1 = a_0 \in K \cap \Gamma_V(K) = K^\times$. □

Now we define $-^T$ to be the identity on V, and extend it to an anti-automorphism of $C_V(K)$ ("anti" means that $(ab)^T = b^T a^T$). Do not confuse $a \mapsto \alpha(a)$ (automorphism), $a \mapsto a^T$ (anti-automorphism), and $a \mapsto \alpha(a^T)$ (anti-automorphism).

Notice that on V, N coincides with the quadratic form N. Many authors seem not to have noticed this, and use different letters. Sometimes they use a sign convention which makes them different.

Now we define the *spinor norm* of $a \in C_V(K)$ by $N(a) = aa^T$. We also define a twisted version: $N^\alpha(a) = a\alpha(a)^T$.

Proposition 23.5.

1. *The restriction of N to $\Gamma_V(K)$ is a homomorphism whose image lies in $K^\times$. N is a mess on the rest of $C_V(K)$.*

2. *The action of $\Gamma_V(K)$ on V is orthogonal. That is, we have a homomorphism $\Gamma_V(K) \to O_V(K)$.*

[4] I promised no Lemmas or Theorems, but I was lying to you.

[5] All these results are true in characteristic 2, but you have to work harder ... you can't go around choosing orthogonal bases because they may not exist.

Proof. First we show that if $a \in \Gamma_V(K)$, then $N^\alpha(a)$ acts trivially on V.

$$
\begin{aligned}
N^\alpha(a)\, v\, \alpha\big(N^\alpha(a)\big)^{-1} &= a\alpha(a)^T v\Big(\alpha(a)\,\underbrace{\alpha\big(\alpha(a)^T\big)}_{=a^T}\Big)^{-1} \\
&= a\,\underbrace{\alpha(a)^T v (a^{-1})^T}_{=(a^{-1}v^T\alpha(a))^T}\,\alpha(a)^{-1} \\
&= aa^{-1}v\alpha(a)\alpha(a)^{-1} \qquad (T|_V = \mathrm{Id}_V \text{ and } a^{-1}v\alpha(a) \in V) \\
&= v
\end{aligned}
$$

So by Lemma 23.4, $N^\alpha(a) \in K^\times$. This implies that N^α is a homomorphism on $\Gamma_V(K)$ because

$$
\begin{aligned}
N^\alpha(a)N^\alpha(b) &= a\alpha(a)^T N^\alpha(b) \\
&= aN^\alpha(b)\alpha(a)^T \qquad\qquad (N^\alpha(b) \text{ is central}) \\
&= ab\alpha(b)^T\alpha(a)^T \\
&= (ab)\alpha(ab)^T = N^\alpha(ab).
\end{aligned}
$$

After all this work with N^α, what we're really interested is N. On the even elements of $\Gamma_V(K)$, N agrees with N^α, and on the odd elements, $N = -N^\alpha$. Since $\Gamma_V(K)$ consists of homogeneous elements, N is also a homomorphism from $\Gamma_V(K)$ to $K^\times$. This proves the first statement of the Proposition.

Finally, since N is a homomorphism on $\Gamma_V(K)$, the action on V preserves the quadratic form $N|_V$. Thus, we have a homomorphism $\Gamma_V(K) \to O_V(K)$. ∎

Now let's analyze the homomorphism $\Gamma_V(K) \to O_V(K)$. Lemma 23.4 says exactly that the kernel is $K^\times$. Next we will show that the image is all of $O_V(K)$. Say $r \in V$ and $N(r) \neq 0$.

$$
\begin{aligned}
rv\alpha(r)^{-1} &= -rv\frac{r}{N(r)} = v - \frac{vr^2 + rvr}{N(r)} \\
&= v - \frac{(v,r)}{N(r)}r \qquad\qquad (23.6) \\
&= \begin{cases} -r & \text{if } v = r \\ v & \text{if } (v,r) = 0 \end{cases} \qquad\qquad (23.7)
\end{aligned}
$$

Thus, r is in $\Gamma_V(K)$, and it acts on V by reflection through the hyperplane $r^\perp$. One might deduce that the homomorphism $\Gamma_V(K) \to O_V(K)$ is surjective because $O_V(K)$ is generated by reflections. This is wrong; $O_V(K)$ is *not* always generated by reflections!

▶ **Exercise 23.1.** Let $H = \mathbb{F}_2^2$, with the quadratic form $x^2 + y^2 + xy$, and let $V = H \oplus H$. Prove that $O_V(\mathbb{F}_2)$ is not generated by reflections.

Remark 23.8. It turns out that this is the *only* counterexample. For any other vector space and/or any other non-degenerate quadratic form, $O_V(K)$ is generated by reflections. The map $\Gamma_V(K) \to O_V(K)$ is surjective even in the example above. Also, in every case except the example above, $\Gamma_V(K)$ is generated as a group by non-zero elements of V (i.e. every element of $\Gamma_V(K)$ is a monomial).

Remark 23.9. Equation 23.6 is the definition of the reflection of v through r. It is only possible to reflect through vectors of non-zero norm. Reflections in characteristic 2 are strange; strange enough that people don't call them reflections, they call them *transvections*.

Thus, we have the diagram

$$\begin{tikzcd}
& 1 \arrow[r] & K^\times \arrow[r] \arrow[d, equal] & \Gamma_V(K) \arrow[r] \arrow[d, "N"] & O_V(K) \arrow[r] \arrow[d, dotted, "N"] & 1 \\
1 \arrow[r] & \pm 1 \arrow[r] & K^\times \arrow[r, "x\mapsto x^2"] & K^\times \arrow[r] & K^\times/(K^\times)^2 \arrow[r] & 1
\end{tikzcd} \tag{23.10}$$

where the rows are exact, $K^\times$ is in the center of $\Gamma_V(K)$ (this is obvious, since $K^\times$ is in the center of $C_V(K)$), and $N : O_V(K) \to K^\times/(K^\times)^2$ is the unique homomorphism sending reflection through $r^\perp$ to $N(r)$ modulo $(K^\times)^2$.

Definition 23.11. $\mathrm{Pin}_V(K) = \{x \in \Gamma_V(K) | N(x) = 1\}$, and $\mathrm{Spin}_V(K) = \mathrm{Pin}_V^0(K)$, the even elements of $\mathrm{Pin}_V(K)$.

On $K^\times$, the spinor norm is given by $x \mapsto x^2$, so the elements of spinor norm 1 are $= \pm 1$. By restricting the top row of (23.10) to elements of norm 1 and even elements of norm 1, respectively, we get exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \pm 1 \arrow[r] & \mathrm{Pin}_V(K) \arrow[r] & O_V(K) \arrow[r, dotted, "N"] & K^\times/(K^\times)^2
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & \pm 1 \arrow[r] & \mathrm{Spin}_V(K) \arrow[r] & SO_V(K) \arrow[r, dotted, "N"] & K^\times/(K^\times)^2
\end{tikzcd}$$

To see exactness of the top sequence, note that the kernel of ϕ is $K^\times \cap \mathrm{Pin}_V(K) = \pm 1$, and that the image of $\mathrm{Pin}_V(K)$ in $O_V(K)$ is exactly the elements of norm 1. The bottom sequence is similar, except that the image of $\mathrm{Spin}_V(K)$ is not all of $O_V(K)$, it is only $SO_V(K)$; by Remark 23.8, every element of $\Gamma_V(K)$ is a product of elements of V, so every element of $\mathrm{Spin}_V(K)$ is a product of an even number of elements of V. Thus, its image is a product of an even number of reflections, so it is in $SO_V(K)$.

??

These maps are NOT always onto, but there are many important cases when they are, like when V has a positive definite quadratic form. The image is the set of elements of $O_V(K)$ or $SO_V(K)$ which have spinor norm 1 in $K^\times/(K^\times)^2$.

What is $N : O_V(K) \to K^\times/(K^\times)^2$? It is the UNIQUE homomorphism such that $N(a) = N(r)$ if a is reflection in $r^\perp$, and r is a vector of norm $N(r)$.

Example 23.12. Take V to be a positive definite vector space over $\mathbb{R}$. Then N maps to 1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2 = \pm 1$ (because N is positive definite). So the spinor norm on $O_V(\mathbb{R})$ is TRIVIAL.

So if V is positive definite, we get double covers

$$1 \to \pm 1 \to \mathrm{Pin}_V(\mathbb{R}) \to O_V(\mathbb{R}) \to 1$$

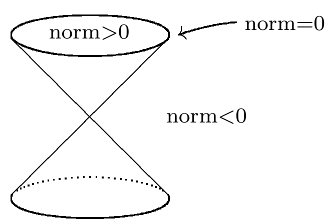
$$1 \to \pm 1 \to \mathrm{Spin}_V(\mathbb{R}) \to SO_V(\mathbb{R}) \to 1$$

This will account for the weird double covers we saw before.

What if V is negative definite. Every reflection now has image -1 in $\mathbb{R}^\times/(\mathbb{R}^\times)^2$, so the spinor norm N is the same as the determinant map $O_V(\mathbb{R}) \to \pm 1$.

So in order to find interesting examples of the spinor norm, you have to look at cases that are neither positive definite nor negative definite.

Let's look at Losrentz space: $\mathbb{R}^{1,3}$.



Reflection through a vector of norm < 0 (spacelike vector, P: parity reversal) has spinor norm -1, det -1 and reflection through a vector of norm > 0 (timelike vector, T: time reversal) has spinor norm $+1$, det -1. So $O_{1,3}(\mathbb{R})$ has 4 components (it is not hard to check that these are all the components), usually called 1, P, T, and PT.

Remark 23.13. For those who know Galois cohomology. We get an exact sequence of algebraic groups

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

(algebraic group means you don't put a field). You do not necessarily get an exact sequence when you put in a field.

If

$$1 \to A \to B \to C \to 1$$

is exact,

$$1 \to A(K) \to B(K) \to C(K)$$

is exact. What you really get is

$$1 \to H^0(\mathrm{Gal}(\bar{K}/K), A) \to H^0(\mathrm{Gal}(\bar{K}/K), B) \to H^0(\mathrm{Gal}(\bar{K}/K), C) \to \\ \to H^1(\mathrm{Gal}(\bar{K}/K), A) \to \cdots$$

It turns out that $H^1(Gal(\bar{K}/K), GL_1) = 1$. However, $H^1(Gal(\bar{K}/K), \pm 1) = K^\times/(K^\times)^2$.

So from

$$1 \to GL_1 \to \Gamma_V \to O_V \to 1$$

you get

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1 = H^1(Gal(\bar{K}/K), GL_1)$$

However, taking

$$1 \to \mu_2 \to \mathrm{Spin}_V \to SO_V \to 1$$

you get

$$1 \to \pm 1 \to \mathrm{Spin}_V(K) \to SO_V(K) \xrightarrow{N} K^\times/(K^\times)^2 = H^1(\bar{K}/K, \mu_2)$$

so the non-surjectivity of N is some kind of higher Galois cohomology.

Warning 23.14. $\mathrm{Spin}_V \to SO_V$ is onto as a map of ALGEBRAIC GROUPS, but $\mathrm{Spin}_V(K) \to SO_V(K)$ need NOT be onto.

Example 23.15. Take $O_3(\mathbb{R}) \cong SO_3(\mathbb{R}) \times \{\pm 1\}$ as 3 is odd (in general $O_{2n+1}(\mathbb{R}) \cong SO_{2n+1}(\mathbb{R}) \times \{\pm 1\}$). So we have a sequence

$$1 \to \pm 1 \to \mathrm{Spin}_3(\mathbb{R}) \to SO_3(\mathbb{R}) \to 1.$$

Notice that $\mathrm{Spin}_3(\mathbb{R}) \subseteq C_3^0(\mathbb{R}) \cong \mathbb{H}$, so $\mathrm{Spin}_3(\mathbb{R}) \subseteq \mathbb{H}^\times$, and in fact we saw that it is S^3.

Lecture 24

Last time we constructed the sequences

$$1 \to K^\times \to \Gamma_V(K) \to O_V(K) \to 1$$

$$1 \to \pm 1 \to \mathrm{Pin}_V(K) \to O_V(K) \xrightarrow{N} K^\times/(K^\times)^2$$

$$1 \to \pm 1 \to \mathrm{Spin}_V(K) \to SO_V(K) \xrightarrow{N} K^\times/(K^\times)^2$$

Spin representations of Spin and Pin groups

Notice that $\mathrm{Pin}_V(K) \subseteq C_V(K)^\times$, so any module over $C_V(K)$ gives a representation of $\mathrm{Pin}_V(K)$. We already figured out that $C_V(K)$ are direct sums of matrix algebras over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

What are the representations (modules) of complex Clifford algebras? Recall that $C_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$, which has a representations of dimension 2^n, which is called the spin representation of $\mathrm{Pin}_V(K)$ and $C_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C}) \times \mathbb{M}_{2^n}(\mathbb{C})$, which has 2 representations, called the spin representations of $\mathrm{Pin}_{2n+1}(K)$.

What happens if we restrict these to $\mathrm{Spin}_V(\mathbb{C}) \subseteq \mathrm{Pin}_V(\mathbb{C})$? To do that, we have to recall that $C^0_{2n}(\mathbb{C}) \cong \mathbb{M}_{2^{n-1}}(\mathbb{C}) \times \mathbb{M}_{2^{n-1}}(\mathbb{C})$ and $C^0_{2n+1}(\mathbb{C}) \cong \mathbb{M}_{2^n}(\mathbb{C})$. So in EVEN dimensions $\mathrm{Pin}_{2n}(\mathbb{C})$ has 1 spin representation of dimension 2^n splitting into 2 HALF SPIN representations of dimension 2^{n-1} and in ODD dimensions, $\mathrm{Pin}_{2n+1}(\mathbb{C})$ has 2 spin representations of dimension 2^n which become the same on restriction to $\mathrm{Spin}_V(\mathbb{C})$.

Now we'll give a second description of spin representations. We'll just do the even dimensional case (odd is similar). Say $\dim V = 2n$, and say we're over $\mathbb{C}$. Choose an orthonormal basis $\gamma_1, \dots, \gamma_{2n}$ for V, so that $\gamma_i^2 = 1$ and $\gamma_i\gamma_j = -\gamma_j\gamma_i$. Now look at the group G generated by $\gamma_1, \dots, \gamma_{2n}$, which is finite, with order 2^{1+2n} (you can write all its elements explicitly). You can see that representations of $C_V(\mathbb{C})$ correspond to representations of G, with -1 acting as -1 (as opposed to acting as 1). So another way to look at representations of the Clifford algebra, you can look at representations of G.

Let's look at the structure of G:

(1) The center is ± 1. This uses the fact that we are in even dimensions, lest $\gamma_1 \cdots \gamma_{2n}$ also be central.

(2) The conjugacy classes: 2 of size 1 (1 and -1), $2^{2n} - 1$ of size 2 ($\pm\gamma_{i_1} \cdots \gamma_{i_n}$), so we have a total of $2^{2n} + 1$ conjugacy classes, so we should have that many representations. G/center is abelian,

isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n}$, which gives us 2^{2n} representations of dimension 1, so there is only one more left to find! We can figure out its dimension by recalling that the sum of the squares of the dimensions of irreducible representations gives us the order of G, which is 2^{2n+1}. So $2^{2n} \times 1^1 + 1 \times d^2 = 2^{2n+1}$, where d is the dimension of the mystery representation. Thus, $d = \pm 2^n$, so $d = 2^n$. Thus, G, and therefore $C_V(\mathbb{C})$, has an irreducible representation of dimension 2^n (as we found earlier in another way).

Example 24.1. Consider $O_{2,1}(\mathbb{R})$. As before, $O_{2,1}(\mathbb{R}) \cong SO_{2,1}(\mathbb{R}) \times (\pm 1)$, and $SO_{2,1}(\mathbb{R})$ is not connected: it has two components, separated by the spinor norm N. We have maps

$$1 \to \pm 1 \to \mathrm{Spin}_{2,1}(\mathbb{R}) \to SO_{2,1}(\mathbb{R}) \xrightarrow{N} \pm 1.$$

$\mathrm{Spin}_{2,1}(\mathbb{R}) \subseteq C^*_{2,1}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R})$, so $\mathrm{Spin}_{2,1}(\mathbb{R})$ has one 2 dimensional spin representation. So there is a map $\mathrm{Spin}_{2,1}(\mathbb{R}) \to SL_2(\mathbb{R})$; by counting dimensions and such, you can show it is an isomorphism. So $\mathrm{Spin}_{2,1}(\mathbb{R}) \cong SL_2(\mathbb{R})$.

Now let's look at some 4 dimensional orthogonal groups

Example 24.2. Look at $SO_4(\mathbb{R})$, which is compact. It has a complex spin representation of dimension $2^{4/2} = 4$, which splits into two half spin representations of dimension 2. We have the sequence

$$1 \to \pm 1 \to \mathrm{Spin}_4(\mathbb{R}) \to SO_4(\mathbb{R}) \to 1 \qquad (N = 1)$$

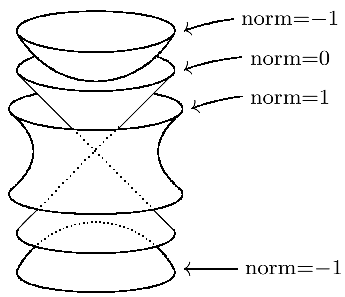
$\mathrm{Spin}_4(\mathbb{R})$ is also compact, so the image in any complex representation is contained in some unitary group. So we get two maps $\mathrm{Spin}_4(\mathbb{R}) \to SU(2) \times SU(2)$, and both sides have dimension 6 and centers of order 4. Thus, we find that $\mathrm{Spin}_4(\mathbb{R}) \cong SU(2) \times SU(2) \cong S^3 \times S^3$, which give you the two half spin representations.

So now we've done the positive definite case.

Example 24.3. Look at $SO_{3,1}(\mathbb{R})$. Notice that $O_{3,1}(\mathbb{R})$ has four components distinguished by the maps $\det, N \to \pm 1$. So we get

$$1 \to \pm 1 \to \mathrm{Spin}_{3,1}(\mathbb{R}) \to SO_{3,1}(\mathbb{R}) \xrightarrow{N} \pm 1 \to 1$$

We expect 2 half spin representations, which give us two homomorphisms $\mathrm{Spin}_{3,1}(\mathbb{R}) \to SL_2(\mathbb{C})$. This time, each of these homomorphisms is an isomorphism (I can't think of why right now). The $SL_2(\mathbb{C})$s are double covers of simple groups. Here, we don't get the splitting into a product as in the positive definite case. This isomorphism is heavily used in quantum field theory because $\mathrm{Spin}_{3,1}(\mathbb{R})$ is a double cover of the connected component of the Lorentz group (and $SL_2(\mathbb{C})$ is easy to work with). Note also that the center of $\mathrm{Spin}_{3,1}(\mathbb{R})$ has order 2, not 4, as for $\mathrm{Spin}_{4,0}(\mathbb{R})$. Also note that the group $PSL_2(\mathbb{C})$ acts on the compactified $\mathbb{C}\cup\{\infty\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) = \frac{a\tau+b}{c\tau+d}$. Subgroups of this group are called KLEINIAN groups. On the other hand, the group $SO_{3,1}(\mathbb{R})^+$ (identity component) acts on $\mathbb{H}^3$ (three dimensional hyperbolic space). To see this, look at



One sheet of norm -1 hyperboloid is isomorphic to $\mathbb{H}^3$ under the induced metric. In fact, we'll define hyperbolic space that way. If you're a topologist, you're very interested in hyperbolic 3-manifolds, which are $\mathbb{H}^3/(\text{discrete subgroup of } SO_{3,1}(\mathbb{R}))$. If you use the fact that $SO_{3,1}(\mathbb{R}) \cong PSL_2(\mathbb{R})$, then you see that these discrete subgroups are in fact Klienian groups.

There are lots of exceptional isomorphisms in small dimension, all of which are very interesting, and almost all of them can be explained by spin groups.

Example 24.4. $O_{2,2}(\mathbb{R})$ has 4 components (given by $\det, N$); $C^0_{2,2}(\mathbb{R}) \cong \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$, which induces an isomorphism $\mathrm{Spin}_{2,2}(\mathbb{R}) \to SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, which give you the two half spin representations. Both sides have dimension 6 with centers of order 4. So this time we get two non-compact groups. Let's look at the fundamental group of $SL_2(\mathbb{R})$, which is $\mathbb{Z}$, so the fundamental group of $\mathrm{Spin}_{2,2}(\mathbb{R})$ is $\mathbb{Z}\oplus\mathbb{Z}$. As we recall, $\mathrm{Spin}_{4,0}(\mathbb{R})$ and $\mathrm{Spin}_{3,1}(\mathbb{R})$ were both simply connected. This shows that SPIN GROUPS NEED NOT BE SIMPLY CONNECTED. So we can take covers of it. What do the corresponding covers (e.g. the universal cover) of $\mathrm{Spin}_{2,2}(\mathbb{R})$ look like? This is hard to describe because for FINITE dimensional complex representations, you get finite dimensional representations of the Lie algebra L, which correspond to the finite dimensional representations of $L\otimes\mathbb{C}$, which correspond to the finite dimensional representations of $L' = $ Lie algebra of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which correspond to the finite dimensional representations of $\mathrm{Spin}_{4,0}(\mathbb{R})$, which has no covers because it is simply connected. This means that any finite dimensional representation of a cover of $\mathrm{Spin}_{2,2}(\mathbb{R})$ actually factors through $\mathrm{Spin}_{2,2}(\mathbb{R})$. So there is no way you can talk about these things with finite matrices, and infinite dimensional representations are hard.

To summarize, the ALGEBRAIC GROUP $\mathrm{Spin}_{2,2}$ is simply connected (as an algebraic group) (think of an algebraic group as a functor from rings to groups), which means that it has no algebraic central extensions. However, the LIE GROUP $\mathrm{Spin}_{2,2}(\mathbb{R})$ is NOT simply connected; it has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. This problem does not happen for COMPACT Lie groups (where every finite cover is algebraic).

We've done $O_{4,0}, O_{3,1}$, and $O_{2,2}$, from which we can obviously get $O_{1,3}$ and $O_{0,4}$. Note that $O_{4,0}(\mathbb{R}) \cong O_{0,4}(\mathbb{R})$, $SO_{4,0}(\mathbb{R}) \cong SO_{0,4}(\mathbb{R})$, $\mathrm{Spin}_{4,0}(\mathbb{R}) \cong \mathrm{Spin}_{0,4}(\mathbb{R})$. However, $\mathrm{Pin}_{4,0}(\mathbb{R}) \not\cong \mathrm{Pin}_{0,4}(\mathbb{R})$. These two are hard to distinguish. We have

$$\begin{tikzcd} \mathrm{Pin}_{4,0}(\mathbb{R}) \arrow[d] & \mathrm{Pin}_{0,4}(\mathbb{R}) \arrow[d] \\ O_{4,0}(\mathbb{R}) \arrow[r, phantom, "="] & O_{0,4}(\mathbb{R}) \end{tikzcd}$$

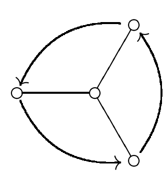
Take a reflection (of order 2) in $O_{4,0}(\mathbb{R})$, and lift it to the Pin groups. What is the order of the lift? The reflection vector v, with $v^2 = \pm 1$ lifts to the element $v \in \Gamma_V(\mathbb{R}) \subseteq C_V^*(\mathbb{R})$. Notice that $v^2 = 1$ in the case of $\mathbb{R}^{4,0}$ and $v^2 = -1$ in the case of $\mathbb{R}^{0,4}$, so in $\mathrm{Pin}_{4,0}(\mathbb{R})$, the reflection lifts to something of order 2, but in $\mathrm{Pin}_{0,4}(\mathbb{R})$, you get an element of order 4!. So these two groups are different.

Two groups are *isoclinic* if they are confusingly similar. A similar phenomenon is common for groups of the form $2 \cdot G \cdot 2$, which means it has a center of order 2, then some group G, and the abelianization has order 2. Watch out.

▶ **Exercise 24.1.** $\mathrm{Spin}_{3,3}(\mathbb{R}) \cong SL_4(\mathbb{R})$.

Triality

This is a special property of 8 dimensional orthogonal groups. Recall that $O_8(\mathbb{C})$ has the Dynkin diagram D_4, which has a symmetry of order three:



But $O_8(\mathbb{C})$ and $SO_8(\mathbb{C})$ do NOT have corresponding symmetries of order three. The thing that does have the symmetry of order three is the spin group! The group $\mathrm{Spin}_8(\mathbb{R})$ DOES have "extra" order three symmetry. You can see it as follows. Look at the half spin representations of $\mathrm{Spin}_8(\mathbb{R})$. Since this is a spin group in even dimension, there are two. $C_{8,0}(\mathbb{R}) \cong \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \times \mathbb{M}_{2^{8/2-1}}(\mathbb{R}) \cong \mathbb{M}_8(\mathbb{R}) \times \mathbb{M}_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has two 8 dimensional real half spin representations. But the spin group is compact, so it preserves some quadratic form, so you get 2 homomorphisms $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$. So $\mathrm{Spin}_8(\mathbb{R})$ has THREE 8 dimensional representations: the half spins, and the one from the map to $SO_8(\mathbb{R})$. These maps $\mathrm{Spin}_8(\mathbb{R}) \to SO_8(\mathbb{R})$ lift to Triality automorphisms $\mathrm{Spin}_8(\mathbb{R}) \to \mathrm{Spin}_8(\mathbb{R})$. The center of $\mathrm{Spin}_8(\mathbb{R})$ is $(\mathbb{Z}/2) + (\mathbb{Z}/2)$ because the center of the Clifford group is $\pm 1, \pm\gamma_1 \cdots \gamma_8$. There are 3 non-trivial elements of the center, and quotienting by any of these gives you something isomorphic to $SO_8(\mathbb{R})$. This is special to 8 dimensions.

More about Orthogonal groups

Is $O_V(K)$ a simple group? NO, for the following reasons:

(1) There is a determinant map $O_V(K) \to \pm 1$, which is usually onto, so it can't be simple.

(2) There is a spinor norm map $O_V(K) \to K^\times/(K^\times)^2$

(3) $-1 \in$ center of $O_V(K)$.

(4) $SO_V(K)$ tends to split if $\dim V = 4$, abelian if $\dim V = 2$, and trivial if $\dim V = 1$.

It turns out that they are usually simple apart from these four reasons why they're not. Let's mod out by the determinant, to get to SO, then look at $\mathrm{Spin}_V(K)$, then quotient by the center, and assume that $\dim V \geq 5$. Then this is usually simple. The center tends to have order 1,2, or 4. If K is a FINITE field, then this gives many finite simple groups.

Note that $SO_V(K)$ is NOT a subgroup of $O_V(K)$, elements of determinant 1 in general, it is the image of $\Gamma_V^0(K) \subseteq \Gamma_V(K) \to O_V(K)$, which is the correct definition. Let's look at why this is right and the definition you know is wrong. There is a homomorphism $\Gamma_V(K) \to \mathbb{Z}/2\mathbb{Z}$, which takes $\Gamma_V^0(K)$ to 0 and $\Gamma_V^1(K)$ to 1 (called the DICKSON INVARIANT). It is easy to check that $\det(v) = (-1)^{\text{dickson invariant}(v)}$. So if the characteristic of K is not 2, $\det = 1$ is equivalent to dickson $= 0$, but in characteristic 2, determinant is the wrong invariant (because determinant is always 1).

Special properties of $O_{1,n}(\mathbb{R})$ and $O_{2,n}(\mathbb{R})$. $O_{1,n}(\mathbb{R})$ acts on hyperbolic space $\mathbb{H}^n$, which is a component of norm -1 vectors in $\mathbb{R}^{n,1}$. $O_{2,n}(\mathbb{R})$ acts on the "Hermitian symmetric space" (Hermitian means it has a complex structure, and symmetric means really nice). There are three ways to construct this space:

(1) It is the set of positive definite 2 dimensional subspaces of $\mathbb{R}^{2,n}$

(2) It is the norm 0 vectors ω of $\mathbb{PC}^{2,n}$ with $(\omega, \bar{\omega}) = 0$.

(3) It is the vectors $x + iy \in \mathbb{R}^{1,n-1}$ with $y \in C$, where the cone C is the interior of the norm 0 cone.

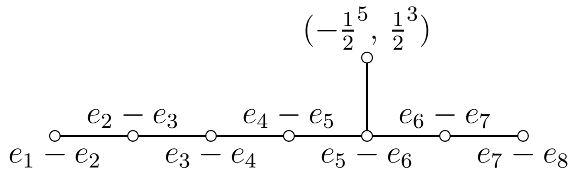
▶ **Exercise 24.2.** Show that these are the same.

Next week, we'll mess around with E_8.

Lecture 25 - E_8

In this lecture we use a vector notation in which powers represent repetitions: so $(1^8) = (1,1,1,1,1,1,1,1)$ and $(\pm\frac{1}{2}^2, 0^6) = (\pm\frac{1}{2}, \pm\frac{1}{2}, 0,0,0,0,0,0)$.

Recall that E_8 has the Dynkin diagram



where each vertex is a root r with $(r,r) = 2$; $(r,s) = 0$ when r and s are not joined, and $(r,s) = -1$ when r and s are joined. We choose an orthonormal basis $e_1, \ldots, e_8$, in which the roots are as given.

We want to figure out what the root lattice L of E_8 is (this is the lattice generated by the roots). If you take $\{e_i - e_{i+1}\} \cup (-1^5,\ 1^3)$ (all the A_7 vectors plus twice the strange vector), they generate the D_8 lattice $= \{(x_1, \ldots, x_8) | x_i \in \mathbb{Z},\quad \sum x_i \text{ even}\}$. So the E_8 lattice consists of two cosets of this lattice, where the other coset is $\{(x_1, \ldots, x_8) | x_i \in \mathbb{Z} + \frac{1}{2},\quad \sum x_i \text{ odd}\}$.

Alternative version: If you reflect this lattice through the hyperplane $e_1^\perp$, then you get the same thing except that $\sum x_i$ is always even. We will freely use both characterizations, depending on which is more convenient for the calculation at hand.

We should also work out the weight lattice, which is the vectors s such that $(r,r)/2$ divides (r,s) for all roots r. Notice that the weight lattice of E_8 is contained in the weight lattice of D_8, which is the union of four cosets of D_8: D_8, $D_8 + (1,\ 0^7)$, $D_8 + (\frac{1}{2}^8)$ and $D_8 + (-\frac{1}{2},\ \frac{1}{2}^7)$. Which of these have integral inner product with the vector $(-\frac{1}{2}^5,\ \frac{1}{2}^3)$? They are the first and the last, so the weight lattice of E_8 is $D_8 \cup D_8 + (-\frac{1}{2}, \frac{1}{2}^7)$, which is equal to the root lattice of E_8.

In other words, the E_8 lattice L is UNIMODULAR (equal to its dual L'), where the dual is the lattice of vectors having integral inner product with all lattice vectors. This is also true of G_2 and F_4, but is not in general true of Lie algebra lattices.

The E_8 lattice is EVEN, which means that the inner product of any vector with itself is always even.

Even unimodular lattices in $\mathbb{R}^n$ only exist if $8|n$ (this 8 is the same 8 that shows up in the periodicity of Clifford groups). The E_8 lattice is the only example in dimension equal to 8 (up to isomorphism, of course). There are two in dimension 16 (one of which is $L \oplus L$, the other is $D_{16} \cup$ some coset). There are 24 in dimension 24, which are the Niemeier lattices. In 32 dimensions, there are more than a billion!

The Weyl group of E_8 is generated by the reflections through $s^\perp$ where $s \in L$ and $(s,s) = 2$ (these are called roots). First, let's find all the roots: $(x_1, \ldots, x_8)$ such that $\sum x_i^2 = 2$ with $x_i \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ and $\sum x_i$ even. If $x_i \in \mathbb{Z}$, obviously the only solutions are permutations of $(\pm 1, \pm 1, 0^6)$, of which there are $\binom{8}{2} \times 2^2 = 112$ choices. In the $\mathbb{Z} + \frac{1}{2}$ case, you can choose the first 7 places to be $\pm\frac{1}{2}$, and the last coordinate is forced, so there are 2^7 choices. Thus, you get 240 roots.

Let's find the orbits of the roots under the action of the Weyl group. We don't yet know what the Weyl group looks like, but we can find a large subgroup that is easy to work with. Let's use the Weyl group of D_8, which consists of the following: we can apply all permutations of the coordinates, or we can change the sign of an even number of coordinates (e.g., reflection in $(1, -1, 0^6)$ swaps the first two coordinates, and reflection in $(1,\ -1,\ 0^6)$ followed by reflection in $(1, 1, 0^6)$ changes the sign of the first two coordinates.)

Notice that under the Weyl group of D_8, the roots form two orbits: the set which is all permutations of $(\pm 1^2, 0^6)$, and the set $(\pm\frac{1}{2}^8)$. Do these become the same orbit under the Weyl group of E_8? Yes; to show this, we just need one element of the Weyl group of E_8 taking some element of the first orbit to the second orbit. Take reflection in $(\frac{1}{2}^8)^\perp$ and apply it to $(1^2, 0^6)$: you get $(\frac{1}{2}^2, -\frac{1}{2}^6)$, which is in the second orbit. So there is just one orbit of roots under the Weyl group.

What do orbits of $W(E_8)$ on other vectors look like? We're interested in this because we might want to do representation theory. The character of a representation is a map from weights to integers, which is $W(E_8)$-invariant. Let's look at vectors of norm 4 for example. So $\sum x_i^2 = 4$, $\sum x_i$ even, and $x_i \in \mathbb{Z}$ or $x_i \in \mathbb{Z} + \frac{1}{2}$. There are 8×2 possibilities which are permutations of $(\pm 2, 0^7)$. There are $\binom{8}{4} \times 2^4$ permutations of $(\pm 1^4, 0^4)$, and there are 8×2^7 permutations of $(\pm\frac{3}{2}, \pm\frac{1}{2}^7)$. So there are a total of 240×9 of these vectors. There are 3 orbits under $W(D_8)$, and as before, they are all one orbit under the action of $W(E_8)$. Just reflect $(2,\ 0^7)$ and $(1^3, -1, 0^4)$ through $(\frac{1}{2}^8)$.

► **Exercise 25.1.** Show that the number of norm 6 vectors is 240×28, and they form one orbit

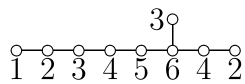
(If you've seen a course on modular forms, you'll know that the number of vectors of norm $2n$ is given by $240 \times \sum_{d|n} d^3$. If you let call these c_n, then $\sum c_n q^n$ is a modular form of level 1 (E_8 even, unimodular), weight 4 ($\dim E_8/2$).)

For norm 8 there are two orbits, because you have vectors that are twice a norm 2 vector, and vectors that aren't. As the norm gets bigger, you'll get a large number of orbits.

What is the order of the Weyl group of E_8? We'll do this by 4 different methods, which illustrate the different techniques for this kind of thing:

(1) This is a good one as a mnemonic. The order of E_8 is given by

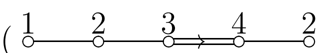
$$|W(E_8)| = 8! \times \prod \left(\begin{array}{c}\text{numbers on the} \\ \text{affine } E_8 \text{ diagram}^{1}\end{array}\right) \times \frac{\text{Weight lattice of } E_8}{\text{Root lattice of } E_8}$$

$$= 8! \times \Big($$


$$\Big) \times 1$$

$$= 2^{14} \times 3^5 \times 5^2 \times 7$$

We can do the same thing for any other Lie algebra, for example,

$$|W(F_4)| = 4! \times ($$


$$) \times 1$$

$$= 2^7 \times 3^2$$

(2) The order of a reflection group is equal to the products of degrees of the fundamental invariants. For E_8, the fundamental invariants are of degrees 2,8,12,14,18,20,24,30 (primes +1).

(3) This one is actually an honest method (without quoting weird facts). The only fact we will use is the following: suppose G acts transitively on a set X with $H =$ the group fixing some point; then $|G| = |H| \cdot |X|$.

This is a general purpose method for working out the orders of groups. First, we need a set acted on by the Weyl group of E_8. Let's take the root vectors (vectors of norm 2). This set has 240 elements, and the Weyl group of E_8 acts transitively on it. So $|W(E_8)| = 240 \times |\text{subgroup fixing } (1, -1, 0^6)|$. But what is the order of this subgroup (call it G_1)? Let's find a set acted on by this group. It acts on the set of norm 2 vectors, but the action is NOT transitive. What are the orbits? G_1 fixes $s = (1, -1, 0^6)$. For other roots r, G_1 obviously fixes (r, s). So how many roots are there with a given inner product with s?

(s, r)	number	choices
2	1	s
1	56	$(1, 0, \pm 1^6), (0, -1, \pm 1^6), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}^6)$
0	126	
-1	56	
-2	1	$-s$

[1] These are the numbers giving highest root.

So there are at least 5 orbits under G_1. In fact, each of these sets is a single orbit under G_1. How can we see this? Find a large subgroup of G_1. Take $W(D_6)$, which is all permutations of the last 6 coordinates and all even sign changes of the last 6 coordinates. It is generated by reflections associated to the roots orthogonal to e_1 and e_2 (those that start with two 0s). The three cases with inner product 1 are three orbits under $W(D_6)$. To see that there is a single orbit under G_1, we just need some reflections that mess up these orbits. If you take a vector $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^6)$ and reflect norm 2 vectors through it, you will get exactly 5 orbits. So G_1 acts transitively on these orbits.

We'll use the orbit of vectors r with $(r, s) = -1$. Let G_2 be the vectors fixing s and r: $\overset{s}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{r}{\circ}$ We have that $|G_1| = |G_2| \cdot 56$.

Keep going ... it gets tedious, but here are the answers up to the last step:

Our plan is to chose vectors acted on by G_i, fixed by G_{i+1} which give us the Dynkin diagram of E_8. So the next step is to try to find vectors t that give us the picture $\overset{s}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{r}{\circ}\cdots\cdots\overset{t}{\circ}$, i.e, they have inner product -1 with r and 0 with s. The possibilities for t are $(-1, -1, 0, 0^5)$ (one of these), $(0, 0, 1, \pm 1, 0^4)$ and permutations of its last five coordinates (10 of these), and $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^5)$ (there are 16 of these), so we get 27 total. Then we could check that they form one orbit, which is boring.

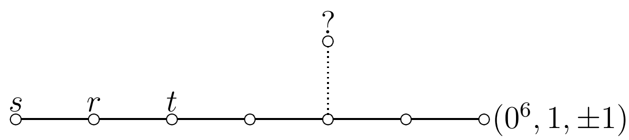
Next find vectors which go next to t in our picture:
$\overset{s}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{r}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\circ}\cdots\cdots\circ$, i.e., whose inner product is -1 with t and zero with r, s. The possibilities are permutations of the last four coords of $(0, 0, 0, 1, \pm 1, 0^3)$ (8 of these) and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^4)$ (8 of these), so there are 16 total. Again check transitivity.

Find a fifth vector; the possibilities are $(0^4, 1, \pm 1, 0^2)$ and perms of the last three coords (6 of these), and $(-\frac{1}{2}^4, \frac{1}{2}, \pm\frac{1}{2}^3)$ (4 of these) for a total of 10.

For the sixth vector, we can have $(0^5, 1, \pm 1, 0)$ or $(0^5, 1, 0, \pm 1)$ (4 possibilites) or $(-\frac{1}{2}^5, \frac{1}{2}, \pm\frac{1}{2}^2)$ (2 possibilities), so we get 6 total.

NEXT CASE IS TRICKY: finding the seventh one, the possibilities are $(0^6, 1, \pm 1)$ (2 of these) and $((-\frac{1}{2})^6, \frac{1}{2}, \frac{1}{2})$ (just 1). The proof of transitivity fails at this point. The group we're using by now doesn't even act transitively on the pair (you can't get between them by changing an even number of signs). What elements of $W(E_8)$ fix all of these first 6 points $\overset{s}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{r}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$
? We want to find roots perpendicular to all of these vectors, and

the only possibility is $((\frac{1}{2})^8)$. How does reflection in this root act on the three vectors above? $(0^6, 1^2) \mapsto ((-\frac{1}{2})^6, \frac{1}{2}^2)$ and $(0^6, 1, -1)$ maps to itself. Is this last vector in the same orbit? In fact they are in different orbits. To see this, look for vectors



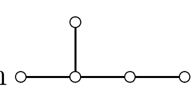
completing the E_8 diagram. In the $(0^6, 1, 1)$ case, you can take the vector $((-\frac{1}{2})^5, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. But in the other case, you can show that there are no possibilities. So these really are different orbits.

Use the orbit with 2 elements, and you get

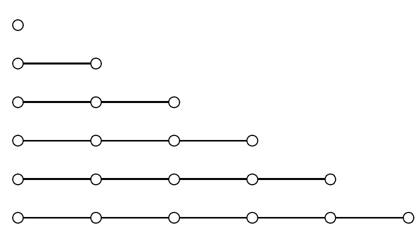
$$|W(E_8)| = 240 \times \underbrace{56 \times \overbrace{27 \times 16 \times 10 \times 6 \times 2 \times 1}^{\text{order of } W(E_6)}}_{\text{order of } W(E_7)}$$

because the group fixing all 8 vectors must be trivial. You also get that

$$|W(\text{“}E_5\text{”})| = 16 \times \underbrace{10 \times \overbrace{6 \times 2 \times 1}^{|W(A_2 \times A_1)|}}_{|W(A_4)|}$$

where “E_5” is the algebra with diagram  (that is, D_5). Similarly, E_4 is A_4 and E_3 is $A_2 \times A_1$.

We got some other information. We found that the Weyl group of E_8 acts transitively on all the configurations



but not on

(4) We'll slip this in to next lecture

Also, next time we'll construct the Lie algebra E_8.

Lecture 26

Today we'll finish looking at $W(E_8)$, then we'll construct E_8.

Remember that we still have a fourth method of finding the order of $W(E_8)$. Let L be the E_8 lattice. Look at $L/2L$, which has 256 elements. Look at this as a set acted on by $W(E_8)$. There is an orbit of size 1 (represented by 0). There is an orbit of size $240/2 = 120$, which are the roots (a root is congruent mod $2L$ to it's negative). Left over are 135 elements. Let's look at norm 4 vectors. Each norm 4 vector, r, satisfies $r \equiv -r \mod 2$, and there are $240 \cdot 9$ of them, which is a lot, so norm 4 vectors must be congruent to a bunch of stuff. Let's look at $r = (2, 0, 0, 0, 0, 0, 0, 0)$. Notice that it is congruent to vectors of the form $(0 \cdots \pm 2 \ldots 0)$, of which there are 16. It is easy to check that these are the only norm 4 vectors congruent to r mod 2. So we can partition the norm 4 vectors into $240 \cdot 9/16 = 135$ subsets of 16 elements. So $L/2L$ has 1+120+135 elements, where 1 is the zero, 120 is represented by 2 elements of norm 2, and 135 is represented by 16 elements of norm 4. A set of 16 elements of norm 4 which are all congruent is called a FRAME. It consists of elements $\pm e_1, \ldots, \pm e_8$, where $e_i^2 = 4$ and $(e_i, e_j) = 1$ for $i \neq j$, so up to sign it is an orthogonal basis.

Then we have

$$|W(E_8)| = (\#\text{ frames}) \times |\text{subgroup fixing a frame}|$$

because we know that $W(E_8)$ acts transitively on frames. So we need to know what the automorphisms of an orthogonal base are. A frame is 8 subsets of the form $(r, -r)$, and isometries of a frame form the group $(\mathbb{Z}/2\mathbb{Z})^8 \cdot S_8$, but these are not all in the Weyl group. In the Weyl group, we found a $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, where the first part is the group of sign changes of an EVEN number of coordinates. So the subgroup fixing a frame must be in between these two groups, and since these groups differ by a factor of 2, it must be one of them. Observe that changing an odd number of signs doesn't preserve the E_8 lattice, so it must be the group $(\mathbb{Z}/2\mathbb{Z})^7 \cdot S_8$, which has order $2^7 \cdot 8!$. So the order of the Weyl group is

$$135 \cdot 2^7 \cdot 8! = |2^7 \cdot S_8| \times \frac{\#\text{ norm 4 elements}}{2 \times \dim L}$$

Remark 26.1. Similarly, if Λ is the Leech lattice, you actually get the order of Conway's group to be

$$|2^{12} \cdot M_{24}| \cdot \frac{\#\text{ norm 8 elements}}{2 \times \dim \Lambda}$$

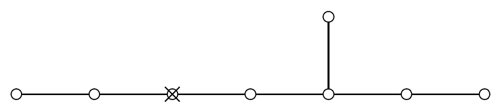
where M_{24} is the Mathieu group (one of the sporadic simple groups). The Leech lattice seems very much to be trying to be the root lattice of the monster group, or something like that. There are a lot of analogies, but nobody can make sense of it.

$W(E_8)$ acts on $(\mathbb{Z}/2\mathbb{Z})^8$, which is a vector space over $\mathbb{F}_2$, with quadratic form $N(a) = \frac{(a,a)}{2} \mod 2$, so you get a map

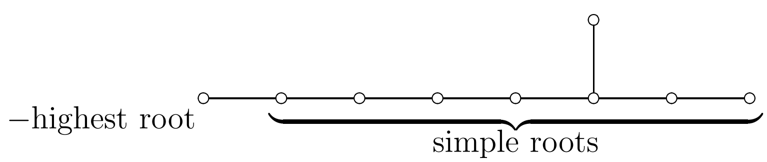
$$\pm 1 \to W(E_8) \to O_8^+(\mathbb{F}_2)$$

which has kernel ± 1 and is surjective. O_8^+ is one of the 8 dimensional orthogonal groups over $\mathbb{F}_2$. So the Weyl group is very close to being an orthogonal group of a vector space over $\mathbb{F}_2$.

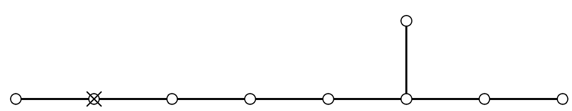
What is inside the root lattice/Lie algebra/Lie group E_8? One obvious way to find things inside is to cover nodes of the E_8 diagram:



If we remove the shown node, you see that E_8 contains $A_2 \times D_5$. We can do better by showing that we can embed the affine $\tilde{E}_8$ in the E_8 lattice.

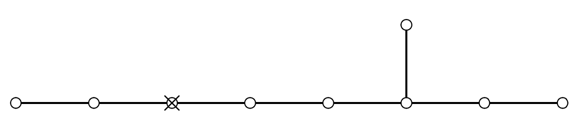


Now you can remove nodes here and get some bigger sub-diagrams. For example, if we cover



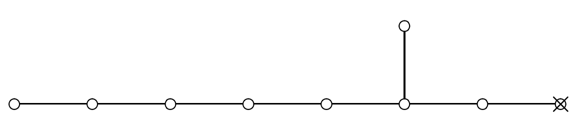
you get that an $A_1 \times E_7$ in E_8. The E_7 consisted of 126 roots orthogonal to a given root. This gives an easy construction of E_7 root system, as all the elements of the E_8 lattice perpendicular to $(1, -1, 0 \dots)$

We can cover



Then we get an $A_2 \times E_6$, where the E_6 are all the vectors with the first 3 coordinates equal. So we get the E_6 lattice for free too.

If you cover



you see that there is a D_8 in E_8, which is all vectors of the E_8 lattice with integer coordinates. We sort of constructed the E_8 lattice this way in the first place.

We can ask questions like: What is the E_8 Lie algebra as a representation of D_8? To answer this, we look at the weights of the E_8 algebra, considered as a module over D_8, which are the 112 roots of the form $(\cdots \pm 1 \cdots \pm 1 \dots)$ and the 128 roots of the form $(\pm 1/2, \dots)$ and 1 vector 0, with multiplicity 8. These give you the Lie algebra of D_8. Recall that D_8 is the Lie algebra of SO_{16}. The double cover has a half spin representation of dimension $2^{16/2-1} = 128$. So E_8 decomposes as a representation of D_8 as the adjoint representation (of dimension 120) plus a half spin representation of dimension 128. This is often used to construct the Lie algebra E_8. We'll do a better construction in a little while.

We've found that the Lie algebra of D_8, which is the Lie algebra of SO_{16}, is contained in the Lie algebra of E_8. Which *group* is contained in the the compact form of the E_8? We found that there were groups

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Spin}_{16}(\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
SO_{16}(\mathbb{R}) & \mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2\mathbb{Z}) & \cong & \mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2\mathbb{Z}) \\
 & PSO_{16}(\mathbb{R}) \arrow[ul, dash] \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

corresponding to subgroups of the center $(\mathbb{Z}/2\mathbb{Z})^2$:

$$\begin{tikzcd}
 & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
 & (\mathbb{Z}/2\mathbb{Z})^2 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

We have a homomorphism $\mathrm{Spin}_{16}(\mathbb{R}) \to E_8(\text{compact})$. What is the kernel? The kernel are elements which act trivially on the Lie algebra of E_8, which is equal to the Lie algebra D_8 plus the half spin representation. On the Lie algebra of D_8, everything in the center is trivial, and on the half spin representation, one of the elements of order 2 is trivial. So the subgroup that you get is the circled one.

▶ **Exercise 26.1.** Show $SU(2) \times E_7(\text{compact})/(-1,-1)$ is a subgroup of E_8 (compact). Similarly, show that $SU(9)/(\mathbb{Z}/3\mathbb{Z})$ is also. These are similar to the example above.

Construction of E_8

Earlier in the course, we had some constructions:

1. using the Serre relations, but you don't really have an idea of what it looks like
2. Take D_8 plus a half spin representation

Today, we'll try to find a natural map from root lattices to Lie algebras. The idea is as follows: Take a basis element e^α (as a formal symbol) for each root α; then take the Lie algebra to be the direct sum of 1 dimensional spaces generated by each e^α and L (L root lattice $\cong$ Cartan subalgebra) . Then we have to define the Lie bracket by setting $[e^\alpha, e^\beta] = e^{\alpha+\beta}$, but then we have a sign problem because $[e^\alpha, e^\beta] \neq -[e^\beta, e^\alpha]$. Is there some way to resolve the sign problem? The answer is that there is no good way to solve this problem (not true, but whatever). Suppose we had a nice functor from root lattices to Lie algebras. Then we would get that the automorphism group of the lattice has to be contained in the automorphism group of the Lie algebra (which is contained in the Lie group), and the automorphism group of the Lattice contains the Weyl group of the lattice. But the Weyl group is NOT usually a subgroup of the Lie group.

We can see this going wrong even in the case of $\mathfrak{sl}_2(\mathbb{R})$. Remember that the Weyl group is $N(T)/T$ where $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ and $N(T) = T \cup \left(\begin{smallmatrix} 0 & b \\ -b^{-1} & 0\end{smallmatrix}\right)$, and this second part is stuff having order 4, so you cannot possibly write this as a semi-direct product of T and the Weyl group.

So the Weyl group is not usually a subgroup of $N(T)$. The best we can do is to find a group of the form $2^n \cdot W \subseteq N(T)$ where n is the rank. For example, let's do it for $SL(n+1, \mathbb{R})$ Then $T = diag(a_1, \dots, a_n)$ with $a_1 \cdots a_n = 1$. Then we take the normalizer of the torus to be $N(T)$ =all permutation matrices with ± 1's with determinant 1, so this is $2^n \cdot S_n$, and it does not split. The problem we had with signs can be traced back to the fact that this group doesn't split.

We can construct the Lie algebra from something acted on by $2^n \cdot W$ (but not from something acted on by W). We take a CENTRAL EXTENSION of the lattice by a group of order 2. Notation is a pain because the lattice is written additively and the extension is nonabelian, so you want it to be written multiplicatively. Write elements of the lattice in the form e^α formally, so we have converted the lattice operation to multiplication. We will use the central extension

$$1 \to \pm 1 \to \hat{e}^L \to \underbrace{e^L}_{\cong L} \to 1$$

We want $\hat{e}^L$ to have the property that $\hat{e}^\alpha \hat{e}^\beta = (-1)^{(\alpha,\beta)} \hat{e}^\beta \hat{e}^\alpha$, where $\hat{e}^\alpha$ is something mapping to e^α. What do the automorphisms of $\hat{e}^L$ look like?

We get

$$1 \to \underbrace{(L/2L)}_{(\mathbb{Z}/2)^{\text{rank}(L)}} \to \text{Aut}(\hat{e}^L) \to \text{Aut}(e^L)$$

for $\alpha \in L/2L$, we get the map $\hat{e}^\beta \to (-1)^{(\alpha,\beta)}\hat{e}^\beta$. The map turns out to be onto, and the group $\text{Aut}(e^L)$ contains the reflection group of the lattice. This extension is usually non-split.

Now the Lie algebra is $L \oplus \{1$ dimensional spaces spanned by $(\hat{e}^\alpha, -\hat{e}^\alpha)\}$ for $\alpha^2 = 2$ with the convention that $-\hat{e}^\alpha$ (-1 in the vector space) is $-\hat{e}^\alpha$ (-1 in the group $\hat{e}^L$). Now define a Lie bracket by the "obvious rules" $[\alpha, \beta] = 0$ for $\alpha, \beta \in L$ (the Cartan subalgebra is abelian), $[\alpha, \hat{e}^\beta] = (\alpha, \beta)\hat{e}^\beta$ ($\hat{e}^\beta$ is in the root space of β), and $[\hat{e}^\alpha, \hat{e}^\beta] = 0$ if $(\alpha, \beta) \geq 0$ (since $(\alpha + \beta)^2 > 2$), $[\hat{e}^\alpha, \hat{e}^\beta] = \hat{e}^\alpha\hat{e}^\beta$ if $(\alpha, \beta) < 0$ (product in the group $\hat{e}^L$), and $[\hat{e}^\alpha, (\hat{e}^\alpha)^{-1}] = \alpha$.

Theorem 26.2. *Assume L is positive definite. Then this Lie bracket forms a Lie algebra (so it is skew and satisfies Jacobi).*

Proof. Easy but tiresome, because there are a lot of cases; let's do them (or most of them).

We check the Jacobi identity: We want $[[a, b], c]+[[b, c], a]+[[c, a], b] = 0$

1. all of a, b, c in L. Trivial because all brackets are zero.

2. two of a, b, c in L. Say α, β, e^γ

$$\underbrace{[[\alpha, \beta], e^\gamma]}_{0} + \underbrace{[[\beta, e^\gamma], \alpha]}_{(\beta,\alpha)(-\alpha,\beta)e^\gamma} + [[e^\gamma, \alpha], \beta]$$

   and similar for the third term, giving a sum of 0.

3. one of a, b, c in L. $\alpha, e^\beta, e^\gamma$. e^β has weight β and e^γ has weight γ and $e^\beta e^\gamma$ has weight $\beta + \gamma$. So check the cases, and you get Jacobi:

$$\begin{aligned}
&[[\alpha, e^\beta], e^\gamma] = (\alpha, \beta)[e^\beta, e^\gamma] \\
&[[e^\beta, e^\gamma], \alpha] = -[\alpha, [e^\beta, e^\gamma]] = -(\alpha, \beta + \gamma)[e^\beta, e^\gamma] \\
&[[e^\gamma, \alpha], e^\beta] = -[[\alpha, e^\gamma], e^\beta] = (\alpha, \gamma)[e^\beta, e^\gamma],
\end{aligned}$$

   so the sum is zero.

4. none of a, b, c in L. This is the really tiresome one, $e^\alpha, e^\beta, e^\gamma$. The main point of going through this is to show that it isn't as tiresome as you might think. You can reduce it to two or three cases. Let's make our cases depending on (α, β), (α, γ), (β, γ).

(a) if 2 of these are 0, then all the $[[*,*],*]$ are zero.

(b) $\alpha = -\beta$. By case a, γ cannot be orthogonal to them, so say $(\alpha,\gamma) = 1$ $(\gamma,\beta) = -1$; adjust so that $e^\alpha e^\beta = 1$, then calculate

$$\begin{aligned}[][e^\gamma, e^\beta], e^\alpha] - [[e^\alpha, e^\beta], e^\gamma] + [[e^\alpha, e^\gamma], e^\beta] &= e^\alpha e^\beta e^\gamma - (\alpha,\gamma)e^\gamma + 0 \\ &= e^\gamma - e^\gamma = 0.\end{aligned}$$

(c) $\alpha = -\beta = \gamma$, easy because $[e^\alpha, e^\gamma] = 0$ and $[[e^\alpha, e^\beta], e^\gamma] = -[[e^\gamma, e^\beta], e^\alpha]$

(d) We have that each of the inner products is 1, 0 or -1. If some $(\alpha,\beta) = 1$, all brackets are 0.

This leaves two cases, which we'll do next time

$\square$

Lecture 27

Last week we talked about $\hat{e}^L$, which was a double cover of e^L. L is the root lattice of E_8. We had the sequence

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$

The Lie algebra structure on $\hat{e}^L$ was given by

$$\begin{aligned}
[\alpha, \beta] &= 0 \\
[\alpha, e^\beta] &= (\alpha, \beta) e^\beta \\
[e^\alpha, e^\beta] &= \begin{cases} 0 & \text{if } (\alpha,\beta) \geq 0 \\ e^\alpha e^\beta & \text{if } (\alpha,\beta) = -1 \\ \alpha & \text{if } (\alpha,\beta) = -2 \end{cases}
\end{aligned}$$

The Lie algebra is $L \oplus \bigoplus_{\alpha^2=2} \hat{e}^\alpha$.

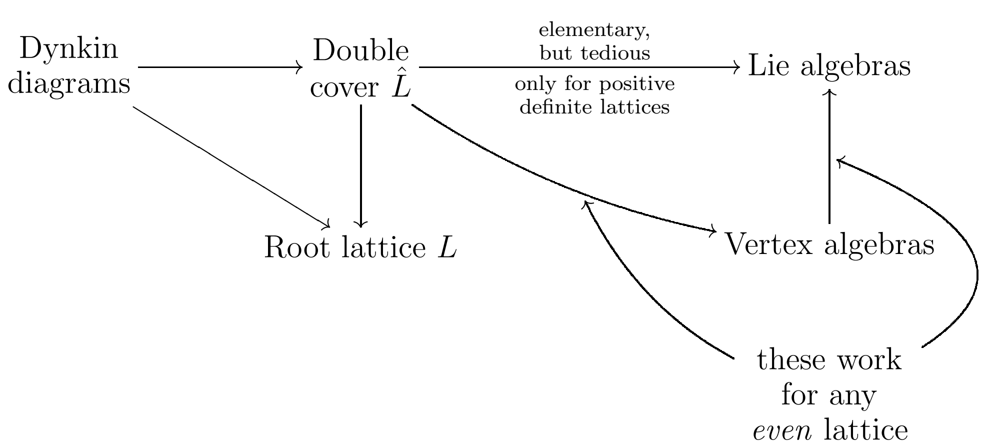
Let's finish checking the Jacobi identity. We had two cases left:

$$[[e^\alpha, e^\beta], e^\gamma] + [[e^\beta, e^\gamma], e^\alpha] + [[e^\gamma, e^\alpha], e^\beta] = 0$$

- $(\alpha,\beta) = (\beta,\gamma) = (\gamma,\alpha) = -1$, in which case $\alpha+\beta+\gamma = 0$. then $[[e^\alpha, e^\beta], e^\gamma] = [e^\alpha e^\beta, e^\gamma] = \alpha+\beta$. By symmetry, the other two terms are $\beta+\gamma$ and $\gamma+\alpha$;the sum of all three terms is $2(\alpha+\beta+\gamma) = 0$.
- $(\alpha,\beta) = (\beta,\gamma) = -1$, $(\alpha,\gamma) = 0$, in which case $[e^\alpha, e^\gamma] = 0$. We check that $[[e^\alpha, e^\beta], e^\alpha] = [e^\alpha e^\beta, e^\gamma] = e^\alpha e^\beta e^\gamma$ (since $(\alpha+\beta,\gamma) = -1$). Similarly, we have $[[e^\beta, e^\gamma], e^\alpha] = [e^\beta e^\gamma, e^\alpha] = e^\beta e^\gamma e^\alpha$. We notice that $e^\alpha e^\beta = -e^\beta e^\alpha$ and $e^\gamma e^\alpha = e^\alpha e^\gamma$ so $e^\alpha e^\beta e^\gamma = -e^\beta e^\gamma e^\alpha$; again, the sum of all three terms in the Jacobi identity is 0.

This concludes the verification of the Jacobi identity, so we have a Lie algebra.

Is there a proof avoiding case-by-case check? Good news: yes! Bad news: it's actually more work. We really have functors as follows:



where $\hat{L}$ is generated by $\hat{e}^{\alpha_i}$ (the i's are the dots in your Dynkin diagram), with $\hat{e}^{\alpha_i}\hat{e}^{\alpha_j} = (-1)^{(\alpha_i,\alpha_j)}\hat{e}^{\alpha_j}\hat{e}^{\alpha_i}$, and -1 is central of order 2.

Unfortunately, you have to spend several weeks learning vertex algebras. In fact, the construction we did was the vertex algebra approach, with all the vertex algebras removed. So there is a more general construction which gives a much larger class of infinite dimensional Lie algebras.

Now we should study the double cover $\hat{L}$, and in particular prove its existence. Given a Dynkin diagram, we can construct $\hat{L}$ as generated by the elements e^{α_i} for α_i simple roots with the given relations. It is easy to check that we get a surjective homomorphism $\hat{L} \to L$ with kernel generated by z with $z^2 = 1$. What's a little harder to show is that $z \neq 1$ (i.e., show that $\hat{L} \neq L$). The easiest way to do it is to use cohomology of groups, but since we have such an explicit case, we'll do it bare hands: Problem: Given Z, H groups with Z abelian, construct central extensions

$$1 \to Z \to G \to H \to 1$$

(where Z lands in the center of G). Let G be the set of pairs (z, h), and set the product $(z_1, h_1)(z_2, h_2) = (z_1z_2c(h_1, h_2), h_1h_2)$, where $c(h_1, h_2) \in Z$ ($c(h_1, h_2)$ will be a cocycle in group cohomology). We obviously get a homomorphism by mapping $(z, h) \mapsto h$. If $c(1, h) = c(h, 1) = 1$ (normalization), then $z \mapsto (z, 1)$ is a homomorphism mapping Z to the center of G. In particular, $(1, 1)$ is the identity. We'll leave it as an exercise to figure out what the inverses are. When is this thing *associative*? Let's just write everything out:

$$\begin{aligned}
\big((z_1, h_1)(z_2, h_2)\big)(z_3, h_3) &= (z_1z_2z_3c(h_1, h_2)c(h_1h_2, h_3), h_1h_2h_3)\\
(z_1, h_1)\big((z_2, h_2)(z_3, h_3)\big) &= (z_1z_2z_3c(h_1, h_2h_3)c(h_2, h_3), h_1h_2h_3)
\end{aligned}$$

so we must have

$$c(h_1, h_2)c(h_1h_2, h_3) = c(h_1h_2, h_3)c(h_2, h_3).$$

This identity is actually very easy to satisfy in one particular case: when c is bimultiplicative: $c(h_1, h_2h_3) = c(h_1, h_2)c(h_1, h_3)$ and $c(h_1h_2, h_3) = c(h_1, h_3)c(h_2, h_3)$. That is, we have a map $H \times H \to Z$. Not all cocycles come from such maps, but this is the case we care about.

To construct the double cover, let $Z = \pm 1$ and $H = L$ (free abelian). If we write H additively, we want c to be a bilinear map $L \times L \to \pm 1$. It is really easy to construct bilinear maps on free abelian groups. Just take any basis $\alpha_1, \ldots, \alpha_n$ of L, choose $c(\alpha_1, \alpha_j)$ arbitrarily for each i, j and extend c via bilinearity to $L \times L$. In our case, we want to find a double cover $\hat{L}$ satisfying $\hat{e}^{\alpha}\hat{e}^{\beta} = (-1)^{(\alpha,\beta)}\hat{e}^{\beta}\hat{e}^{\alpha}$ where $\hat{e}^{\alpha}$ is a lift of e^{α}.

This just means that $c(\alpha,\beta) = (-1)^{(\alpha,\beta)}c(\beta,\alpha)$. To satisfy this, just choose $c(\alpha_i,\alpha_j)$ on the basis $\{\alpha_i\}$ so that $c(\alpha_i,\alpha_j) = (-1)^{(\alpha_i,\alpha_j)}c(\alpha_j,\alpha_i)$. This is trivial to do as $(-1)^{(\alpha_i,\alpha_i)} = 1$. Notice that this uses the fact that the lattice is even. There is no canonical way to choose this 2-cocycle (otherwise, the central extension would split as a product), but all the different double covers are isomorphic because we can specify $\hat{L}$ by generators and relations. Thus, we have constructed $\hat{L}$ (or rather, verified that the kernel of $\hat{L} \to L$ has order 2, not 1).

Let's now look at lifts of automorphisms of L to $\hat{L}$.

► **Exercise 27.1.** Any automorphism of L preserving $(\ ,\ )$ lifts to an automorphism of $\hat{L}$

There are two special cases:

1. -1 is an automorphism of L, and we want to lift it to $\hat{L}$ explicitly. First attempt: try sending $\hat{e}^\alpha$ to $\hat{e}^{-\alpha} := (\hat{e}^\alpha)^{-1}$, which doesn't work because $a \mapsto a^{-1}$ is not an automorphism on non-abelian groups.

   Better: $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$ is an automorphism of $\hat{L}$. To see this, check

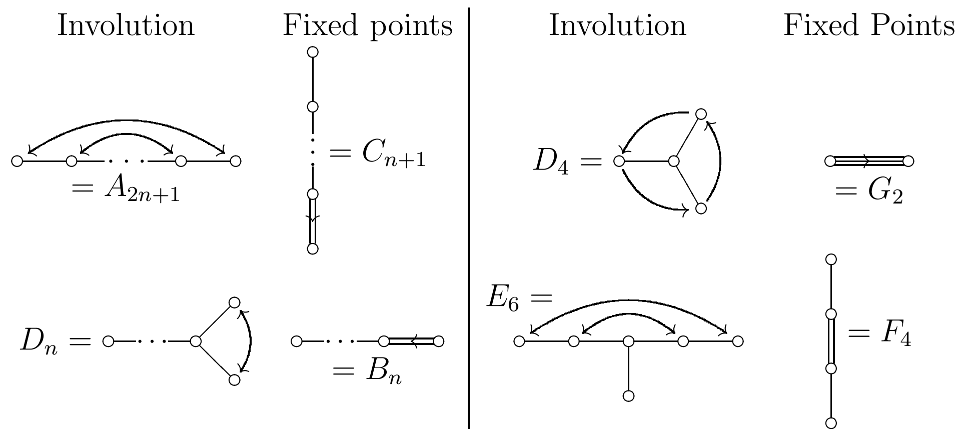
   $$\begin{aligned} \omega(\hat{e}^\alpha)\omega(\hat{e}^\beta) &= (-1)^{(\alpha^2+\beta^2)/2}(\hat{e}^\alpha)^{-1}(\hat{e}^\beta)^{-1} \\ \omega(\hat{e}^\alpha\hat{e}^\beta) &= (-1)^{(\alpha+\beta)^2/2}(\hat{e}^\beta)^{-1}(\hat{e}^\alpha)^{-1} \end{aligned}$$

   which work out just right

2. If $r^2 = 2$, then $\alpha \mapsto \alpha - (\alpha,r)r$ is an automorphism of L (reflection through $r^\perp$). You can lift this by $\hat{e}^\alpha \mapsto \hat{e}^\alpha(\hat{e}^r)^{-(\alpha,r)} \times (-1)^{\binom{(\alpha,r)}{2}}$. This is a homomorphism (check it!) of order (usually) 4!

   Remark 27.1. Although automorphisms of L lift to automorphisms of $\hat{L}$, the lift might have larger order.

This construction works for the root lattices of A_n, D_n, E_6, E_7, and E_8; these are the lattices which are even, positive definite, and generated by vectors of norm 2 (in fact, all such lattices are sums of the given ones). What about B_n, C_n, F_4 and G_2? The reason the construction doesn't work for these cases is because there are roots of different lengths. These all occur as fixed points of diagram automorphisms of A_n, D_n and E_6. In fact, we have a *functor* from Dynkin diagrams to Lie algebras, so and automorphism of the diagram gives an automorphism of the algebra



A_{2n} doesn't really give you a new algebra: it corresponds to some superalgebra stuff.

Construction of the Lie group of E_8

It is the group of automorphisms of the Lie algebra generated by the elements $\exp(\lambda Ad(\hat{e}^\alpha))$, where λ is some real number, $\hat{e}^\alpha$ is one of the basis elements of the Lie algebra corresponding to the root α, and $Ad(\hat{e}^\alpha)(a) = [\hat{e}^\alpha, a]$. In other words,

$$\exp(\lambda Ad(\hat{e}^\alpha))(a) = 1 + \lambda[\hat{e}^\alpha, a] + \frac{\lambda^2}{2}[\hat{e}^\alpha, [\hat{e}^\alpha, a]].$$

and all the higher terms are zero. To see that $Ad(\hat{e}^\alpha)^3 = 0$, note that if β is a root, then $\beta + 3\alpha$ is not a root (or 0).

Warning 27.2. In general, the group generated by these automorphisms is NOT the whole automorphism group of the Lie algebra. There might be extra diagram automorphisms, for example.

We get some other things from this construction. We can get simple groups over finite fields: note that the construction of a Lie algebra above works over any commutative ring (e.g. over $\mathbb{Z}$). The only place we used division is in $\exp(\lambda Ad(\hat{e}^\alpha))$ (where we divided by 2). The only time this term is non-zero is when we apply $\exp(\lambda Ad(\hat{e}^\alpha))$ to $\hat{e}^{-\alpha}$, in which case we find that $[\hat{e}^\alpha, [\hat{e}^\alpha, \hat{e}^{-\alpha}]] = [\hat{e}^\alpha, \alpha] = -(\alpha, \alpha)\hat{e}^\alpha$, and the fact that $(\alpha, \alpha) = 2$ cancels the division by 2. So we can in fact construct the E_8 group over *any* commutative ring. You can mumble something about group schemes over $\mathbb{Z}$ at this point. In particular, we have groups of type E_8 over *finite fields*, which are actually finite simple groups (these are called Chevalley groups; it takes work to show that they are simple, there is a book by Carter called *Finite Simple Groups* which you can look at).

Real forms

So we've constructed the Lie group and Lie algebra of type E_8. There are in fact several *different* groups of type E_8. There is one *complex* Lie algebra of type E_8, which corresponds to several different real Lie algebras of type E_8.

Let's look at some smaller groups:

Example 27.3. $\mathfrak{sl}_2(\mathbb{R}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with a, b, c, d real $a + d = 0$; this is not compact. On the other hand, $\mathfrak{su}_2(\mathbb{R}) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $d = -a$ imaginary $b = -\bar{c}$, is compact. These have the same Lie algebra over $\mathbb{C}$.

Let's look at what happens for E_8. In general, suppose L is a Lie algebra with complexification $L \otimes \mathbb{C}$. How can we find another Lie algebra M with the same complexification? $L \otimes \mathbb{C}$ has an anti-linear involution $\omega_L : l \otimes z \mapsto l \otimes \bar{z}$. Similarly, it has an anti-linear involution ω_M. Notice that $\omega_L \omega_M$ is a linear involution of $L \otimes \mathbb{C}$. Conversely, if we know this involution, we can reconstruct M from it. Given an involution ω of $L \otimes \mathbb{C}$, we can get M as the fixed points of the map $a \mapsto \omega_L \omega(a)$ "=" $\overline{\omega(a)}$. Another way is to put $L = L^+ \oplus L^-$, which are the $+1$ and -1 eigenspaces, then $M = L^+ \oplus iL^-$.

Thus, to find other real forms, we have to study the involutions of the complexification of L. The exact relation is kind of subtle, but this is a good way to go.

Example 27.4. Let $L = \mathfrak{sl}_2(\mathbb{R})$. It has an involution $\omega(m) = -m^T$. $\mathfrak{su}_2(\mathbb{R})$ is the set of fixed points of the involution ω times complex conjugation on $\mathfrak{sl}_2(\mathbb{C})$, by definition.

So to construct real forms of E_8, we want some involutions of the Lie algebra E_8 which we constructed. What involutions do we know about? There are two obvious ways to construct involutions:

1. Lift -1 on L to $\hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}(\hat{e}^\alpha)^{-1}$, which induces an involution on the Lie algebra.
2. Take $\beta \in L/2L$, and look at the involution $\hat{e}^\alpha \mapsto (-1)^{(\alpha,\beta)}\hat{e}^\alpha$.

(2) gives nothing new ... you get the Lie algebra you started with. (1) only gives you one real form. To get all real forms, you multiply these two kinds of involutions together.

Recall that $L/2L$ has 3 orbits under the action of the Weyl group, of size 1, 120, and 135. These will correspond to the three real forms of E_8. How do we distinguish different real forms? The answer was found by Cartan: look at the signature of an invariant quadratic form on the Lie algebra!

A bilinear form (,) on a Lie algebra is called *invariant* if $([a,b],c)+(b[a,c])=0$ for all a,b,c. This is called invariant because it corresponds to the form being invariant under the corresponding group action. Now we can construct an invariant bilinear form on E_8 as follows:

1. $(\alpha,\beta)_{\text{in the Lie algebra}}=(\alpha,\beta)_{\text{in the lattice}}$
2. $(\hat{e}^\alpha,(\hat{e}^\alpha)^{-1})=1$
3. $(a,b)=0$ if a and b are in root spaces α and β with $\alpha+\beta\neq 0$.

This gives an invariant inner product on E_8, which you prove by case-by-case check

▶ **Exercise 27.2.** do these checks

Next time, we'll use this to produce bilinear forms on all the real forms and then we'll calculate the signatures.

Lecture 28

Last time, we constructed a Lie algebra of type E_8, which was $L \oplus \bigoplus \hat{e}^\alpha$, where L is the root lattice and $\alpha^2 = 2$. This gives a double cover of the root lattice:

$$1 \to \pm 1 \to \hat{e}^L \to e^L \to 1.$$

We had a lift for $\omega(\alpha) = -\alpha$, given by $\omega(\hat{e}^\alpha) = (-1)^{(\alpha^2/2)}(\hat{e}^\alpha)^{-1}$. So ω becomes an automorphism of order 2 on the Lie algebra. $e^\alpha \mapsto (-1)^{(\alpha,\beta)} e^\alpha$ is also an automorphism of the Lie algebra.

Suppose σ is an automorphism of order 2 of the real Lie algebra $L = L^+ + L^-$ (eigenspaces of σ). We saw that you can construct another real form given by $L^+ + iL^-$. Thus, we have a map from conjugacy classes of automorphisms with $\sigma^2 = 1$ to real forms of L. This is not in general in isomorphism.

Today we'll construct some more real forms of E_8. E_8 has an invariant symmetric bilinear form $(e^\alpha, (e^\alpha)^{-1}) = 1$, $(\alpha, \beta) = (\beta, \alpha)$. The form is unique up to multiplication by a constant since E_8 is an irreducible representation of E_8. So the *absolute value of the signature* is an invariant of the Lie algebra.

For the split form of E_8, what is the signature of the invariant bilinear form (the split form is the one we just constructed)? On the Cartan subalgebra L, (,) is positive definite, so we get $+8$ contribution to the signature. On $\{e^\alpha, (e^\alpha)^{-1}\}$, the form is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it has signature $0 \cdot 120$. Thus, the signature is 8. So if we find any real form with a different signature, we'll have found a new Lie algebra.

Let's first try involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)} e^\alpha$. But this doesn't change the signature. L is still positive definite, and you still have $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ on the other parts. These Lie algebras actually turn out to be isomorphic to what we started with (though we haven't shown that they are isomorphic).

Now try $\omega : e^\alpha \mapsto (-1)^{\alpha^2/2}(e^\alpha)^{-1}$, $\alpha \mapsto -\alpha$. What is the signature of the form? Let's write down the $+$ and $-$ eigenspaces of ω. The $+$ eigenspace will be spanned by $e^\alpha - e^{-\alpha}$, and these vectors have norm -2 and are orthogonal. The $-$ eigenspace will be spanned by $e^\alpha + e^{-\alpha}$ and L, which have norm 2 and are orthogonal, and L is positive definite. What is the Lie algebra corresponding to the involution ω? It will be spanned by $e^\alpha - e^{-\alpha}$ where $\alpha^2 = 2$ (norm -2), and $i(e^\alpha + e^{-\alpha})$ (norm -2), and iL (which is now negative definite). So the bilinear form is *negative definite*, with signature $-248 (\neq \pm 8)$.

With some more work, you can actually show that this is the Lie algebra of the *compact* form of E_8. This is because the automorphism group of E_8 preserves the invariant bilinear form, so it is contained in $O_{0,248}(\mathbb{R})$, which is compact.

Now let's look at involutions of the form $e^\alpha \mapsto (-1)^{(\alpha,\beta)}\omega(e^\alpha)$. Notice that ω commutes with $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha$. The β's in (α, β) correspond to $L/2L$ modulo the action of the Weyl group $W(E_8)$. Remember this has three orbits, with 1 norm 0 vector, 120 norm 2 vectors, and 135 norm 4 vectors. The norm 0 vector gives us the compact form. Let's look at the other cases and see what we get.

Suppose V has a negative definite symmetric inner product $(\ ,\ )$, and suppose σ is an involution of $V = V_+ \oplus V_-$ (eigenspaces of σ). What is the signature of the invariant inner product on $V_+ \oplus iV_-$? On V_+, it is negative definite, and on iV_- it is positive definite. Thus, the signature is $\dim V_- - \dim V_+ = -\operatorname{tr}(\sigma)$. So we want to work out the traces of these involutions.

Given some $\beta \in L/2L$, what is $\operatorname{tr}(e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^\alpha)$? If $\beta = 0$, the traces is obviously 248 because we just have the identity map. If $\beta^2 = 2$, we need to figure how many roots have a given inner product with β. Recall that this was determined before:

(α, β)	# of roots α with given inner product	eigenvalue
2	1	1
1	56	-1
0	126	1
-1	56	-1
-2	1	1

Thus, the trace is $1 - 56 + 126 - 56 + 1 + 8 = 24$ (the 8 is from the Cartan subalgebra). So the signature of the corresponding form on the Lie algebra is -24. We've found a third Lie algebra.

If we also look at the case when $\beta^2 = 4$, what happens? How many α with $\alpha^2 = 2$ and with given (α, β) are there? In this case, we have:

(α, β)	# of roots α with given inner product	eigenvalue
2	14	1
1	64	-1
0	84	1
-1	64	-1
-2	14	1

The trace will be $14 - 64 + 84 - 64 + 14 + 8 = -8$. This is just the split form again.

Summary: We've found 3 forms of E_8, corresponding to 3 classes in $L/2L$, with signatures 8, -24, -248, corresponding to involutions $e^\alpha \mapsto (-1)^{(\alpha,\beta)}e^{-\alpha}$ of the *compact* form. If L is the *compact* form of a simple Lie algebra, then Cartan showed that the other forms correspond exactly to the conjugacy classes of involutions in the automorphism group of L (this doesn't work if you don't start with the compact form — so always start with the compact form).

In fact, these three are the *only* forms of E_8, but we won't prove that.

Working with simple Lie groups

As an example of how to work with simple Lie groups, we will look at the general question: Given a simple Lie group, what is its homotopy type? Answer: G has a unique conjugacy class of maximal compact subgroups K, and G is homotopy equivalent to K.

Proof for $GL_n(\mathbb{R})$. First pretend $GL_n(\mathbb{R})$ is simple, even though it isn't; whatever. There is an obvious compact subgroup: $O_n(\mathbb{R})$. Suppose K is *any* compact subgroup of $GL_n(\mathbb{R})$. Choose any positive definite form $(\ ,\ )$ on $\mathbb{R}^n$. This will probably not be invariant under K, but since K is compact, we can average it over K get one that is: define a new form $(a,b)_{\text{new}} = \int_K (ka, kb)\, dk$. This gives an invariant positive definite bilinear form (since integral of something positive definite is positive definite). Thus, any compact subgroup preserves some positive definite form. But the subgroup fixing some positive definite bilinear form is conjugate to a subgroup of $O_n(\mathbb{R})$ (to see this, diagonalize the form). So K is contained in a conjugate of $O_n(\mathbb{R})$.

Next we want to show that $G = GL_n(\mathbb{R})$ is homotopy equivalent to $O_n(\mathbb{R}) = K$. We will show that $G = KAN$, where K is O_n, A is all diagonal matrices with positive coefficients, and N is matrices which are upper triangular with 1s on the diagonal. This is the *Iwasawa decomposition*. In general, we get K compact, A semisimple abelian, and N is unipotent. The proof of this you saw before was called the Grahm-Schmidt process for orthonormalizing a basis. Suppose $v_1, \ldots, v_n$ is any basis for $\mathbb{R}^n$.

1. Make it orthogonal by subtracting some stuff, you'll get v_1, $v_2 - *v_1$, $v_3 - *v_2 - *v_1$, $\ldots$.
2. Normalize by multiplying each basis vector so that it has norm 1. Now we have an orthonormal basis.

This is just another way to say that GL_n can be written as KAN. Making things orthogonal is just multiplying by something in N, and normalizing is just multiplication by some diagonal matrix with positive entries. An orthonormal basis is an element of O_n. Tada! This decomposition is just a topological one, not a decomposition as groups. Uniqueness is easy to check.

Now we can get at the homotopy type of GL_n. $N \cong \mathbb{R}^{n(n-1)/2}$, and $A \cong (\mathbb{R}^+)^n$, which are contractible. Thus, $GL_n(\mathbb{R})$ has the same homotopy type as $O_n(\mathbb{R})$, its maximal compact subgroup. □

If you wanted to know $\pi_1(GL_3(\mathbb{R}))$, you could calculate $\pi_1(O_3(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$, so $GL_3(\mathbb{R})$ has a double cover. Nobody has shown you this double cover because it is *not algebraic*.

Example 28.1. Let's go back to various forms of E_8 and figure out (guess) the fundamental groups. We need to know the maximal compact subgroups.

1. One of them is easy: the compact form is its own maximal compact subgroup. What is the fundamental group? Remember or quote the fact that for compact simple groups, $\pi_1 \cong \frac{\text{weight lattice}}{\text{root lattice}}$, which is 1. So this form is simply connected.

2. $\beta^2 = 2$ case (signature -24). Recall that there were 1, 56, 126, 56, and 1 roots α with $(\alpha, \beta) = 2, 1, 0, -1$, and -2 respectively, and there are another 8 dimensions for the Cartan subalgebra. On the $1, 126, 1, 8$ parts, the form is negative definite. The sum of these root spaces gives a Lie algebra of type E_7A_1 with a negative definite bilinear form (the 126 gives you the roots of an E_7, and the 1s are the roots of an A_1). So it a reasonable guess that the maximal compact subgroup has something to do with E_7A_1. E_7 and A_1 are not simply connected: the compact form of E_7 has $\pi_1 = \mathbb{Z}/2$ and the compact form of A_1 also has $\pi_1 = \mathbb{Z}/2$. So the universal cover of E_7A_1 has center $(\mathbb{Z}/2)^2$. Which part of this acts trivially on E_8? We look at the E_8 Lie algebra as a representation of $E_7 \times A_1$. You can read off how it splits form the picture above: $E_8 \cong E_7 \oplus A_1 \oplus 56 \otimes 2$, where 56 and 2 are irreducible, and the centers of E_7 and A_1 both act as -1 on them. So the maximal compact subgroup of this form of E_8 is the simply connected compact form of $E_7 \times A_1/(-1, -1)$. This means that $\pi_1(E_8)$ is the same as π_1 of the compact subgroup, which is $(\mathbb{Z}/2)^2/(-1, -1) \cong \mathbb{Z}/2$. So this simple group has a nontrivial double cover (which is non-algebraic).

3. For the other (split) form of E_8 with signature 8, the maximal compact subgroup is $\mathrm{Spin}_{16}(\mathbb{R})/(\mathbb{Z}/2)$, and $\pi_1(E_8)$ is $\mathbb{Z}/2$.

You can compute any other homotopy invariants with this method.

Let's look at the 56 dimensional representation of E_7 in more detail.

We had the picture

(α, β)	# of α's
2	1
1	56
0	126
-1	56
-2	1

The Lie algebra E_7 fixes these 5 spaces of E_8 of dimensions $1, 56, 126 + 8, 56, 1$. From this we can get some representations of E_7. The $126 + 8$ splits as $1+(126+7)$. But we also get a 56 dimensional representation of E_7. Let's show that this is actually an irreducible representation. Recall that in calculating $W(E_8)$, we showed that $W(E_7)$ acts transitively on this set of 56 roots of E_8, which can be considered as weights of E_7.

An irreducible representation is called *minuscule* if the Weyl group acts transitively on the weights. This kind of representation is particularly easy to work with. It is really easy to work out the character for example: just translate the 1 at the highest weight around, so every weight has multiplicity 1.

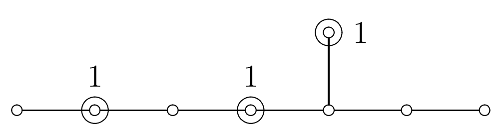
So the 56 dimensional representation of E_7 must actually be the irreducible representation with whatever highest weight corresponds to one of the vectors.

Every possible simple Lie group

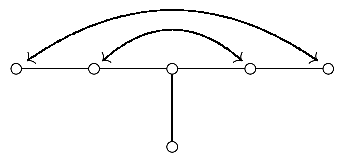
We will construct them as follows: Take an involution σ of the compact form $L = L^+ + L^-$ of the Lie algebra, and form $L^+ + iL^-$. The way we constructed these was to first construct A_n, D_n, E_6, and E_7 as for E_8. Then construct the involution $\omega : e^\alpha \mapsto -e^{-\alpha}$. We get B_n, C_n, F_4, and G_2 as fixed points of the involution ω.

Kac classified all automorphisms of finite order of any compact simple Lie group. The method we'll use to classify involutions is extracted from his method. We can construct lots of involutions as follows:

1. Take any Dynkin diagram, say E_8, and select some of its vertices, corresponding to simple roots. Get an involution by taking $e^\alpha \mapsto \pm e^\alpha$ where the sign depends on whether α is one of the simple roots we've selected. However, this is not a great method. For one thing, you get a lot of repeats (recall that there are only 3, and we've found 2^8 this way).



2. Take any diagram automorphism of order 2, such as



This gives you more involutions.

Next time, we'll see how to cut down this set of involutions.

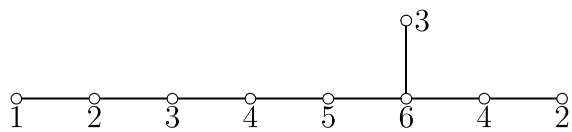
Lecture 29

Split form of Lie algebra (we did this for A_n, D_n, E_6, E_7, E_8): $A = \bigoplus \hat{e}^\alpha \oplus L$. Compact form $A^+ + iA^-$, where $A^\pm$ eigenspaces of $\omega : \hat{e}^\alpha \mapsto (-1)^{\alpha^2/2}\hat{e}^{-\alpha}$.

We talked about other involutions of the compact form. You get all the other forms this way.

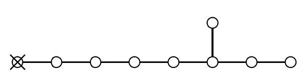
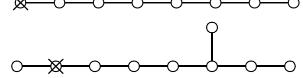
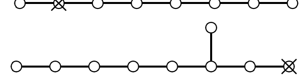
The idea now is to find ALL real simple Lie algebras by listing all involutions of the compact form. We will construct all of them, but we won't prove that we have all of them.

We'll use Kac's method for classifying all automorphisms of order N of a compact Lie algebra (and we'll only use the case $N = 2$). First let's look at inner automorphisms. Write down the AFFINE Dynkin diagram

$-$highest weight = 

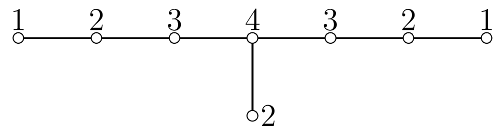
Choose n_i with $\sum n_i m_i = N$ where the m_i are the numbers on the diagram. We have an automorphism $e^{\alpha_j} \mapsto e^{2\pi i n_j/N} e^{\alpha_j}$ induces an automorphism of order dividing N. This is obvious. The point of Kac's theorem is that all inner automorphisms of order dividing N are obtained this way and are conjugate if and only if they are conjugate by an automorphism of the Dynkin diagram. We won't actually prove Kac's theorem because we just want to get a bunch of examples. See [Kac90] or [Hel01].

Example 29.1. Real forms of E_8. We've already found three, and it took us a long time. We can now do it fast. We need to solve $\sum n_i m_i = 2$ where $n_i \geq 0$; there are only a few possibilities:

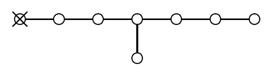
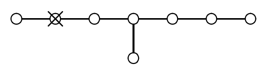
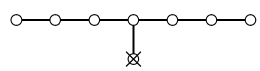
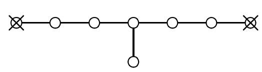
$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way		E_8 (compact form)
1×2	two ways		$A_1 E_7$
			D_8 (split form)
$1 \times 1 + 1 \times 1$	no ways		

The points NOT crossed off form the Dynkin diagram of the maximal compact subgroup. Thus, by just looking at the diagram, we can see what all the real forms are!

Example 29.2. Let's do E_7. Write down the affine diagram:



We get the possibilities

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way*		E_7 (compact form)
1×2	two ways*		$A_1 D_6$
			A_7 (split form)**
$1 \times 1 + 1 \times 1$	one way		$E_6 \oplus \mathbb{R}$ ***

(*) The number of ways is counted up to automorphisms of the diagram.
(**) In the split real form, the maximal compact subgroup has dimension equal to half the number of roots. The roots of A_7 look like $\varepsilon_i - \varepsilon_j$ for $i, j \leq 8$ and $i \neq j$, so the dimension is $8 \cdot 7 + 7 = 56 = \frac{112}{2}$.
(***) The maximal compact subgroup is $E_6 \oplus \mathbb{R}$ because the fixed subalgebra contains the whole Cartan subalgebra, and the E_6 only accounts for 6 of the 7 dimensions. You can use this to construct some interesting representations of E_6 (the minuscule ones). How does the algebra E_7 decompose as a representation of the algebra $E_6 \oplus \mathbb{R}$?

We can decompose it according to the eigenvalues of $\mathbb{R}$. The $E_6 \oplus \mathbb{R}$ is the zero eigenvalue of $\mathbb{R}$ [why?], and the rest is 54 dimensional. The easy way to see the decomposition is to look at the roots. Remember when we computed the Weyl group we looked for vectors like

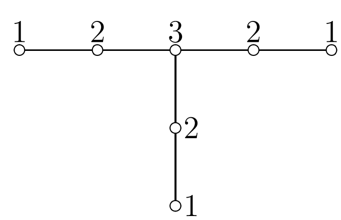
 or

The 27 possibilities (for each) form the weights of a 27 dimensional representation of E_6. The orthogonal complement of the two nodes is an E_6 root system whose Weyl group acts transitively on these 27 vectors (we showed that these form a single orbit, remember?). Vectors of the E_7 root system are the vectors of the E_6 root system plus these 27 vectors plus the other 27 vectors. This splits up the E_7 explicitly. The two 27s form single orbits, so they are irreducible. Thus, $E_7 \cong E_6 \oplus \mathbb{R} \oplus 27 \oplus 27$, and the 27s are minuscule.

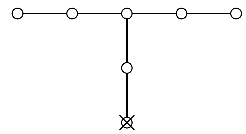
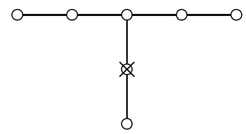
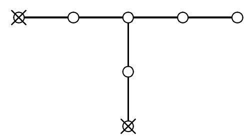
Let K be a maximal compact subgroup, with Lie algebra $\mathbb{R} + E_6$. The factor of $\mathbb{R}$ means that K has an S^1 in its center. Now look at the space G/K, where G is the Lie group of type E_7, and K is the maximal compact subgroup. It is a *Hermitian symmetric space*. Symmetric space means that it is a (simply connected) Riemannian manifold M such that for each point $p \in M$, there is an automorphism fixing p and acting as -1 on the tangent space. This looks weird, but it turns out that all kinds of nice objects you know about are symmetric spaces. Typical examples you may have seen: spheres S^n, hyperbolic space $\mathbb{H}^n$, and Euclidean space $\mathbb{R}^n$. Roughly speaking, symmetric spaces have nice properties of these spaces. Cartan classified all symmetric spaces: they are non-compact simple Lie groups modulo the maximal compact subgroup (more or less ... depending on simply connectedness hypotheses 'n such). Historically, Cartan classified simple Lie groups, and then later classified symmetric spaces, and was surprised to find the same result. Hermitian symmetric spaces are just symmetric spaces with a complex structure. A standard example of this is the upper half plane $\{x + iy | y > 0\}$. It is acted on by $SL_2(\mathbb{R})$, which acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$.

Let's go back to this G/K and try to explain why we get a Hermitian symmetric space from it. We'll be rather sketchy here. First of all, to make it a symmetric space, we have to find a nice invariant Riemannian metric on it. It is sufficient to find a positive definite bilinear form on the tangent space at p which is invariant under K ... then you can translate it around. We can do this as K is compact (so you have the averaging trick). Why is it Hermitian? We'll show that there is an almost complex structure. We have S^1 acting on the tangent space of each point because we have an S^1 in the center of the stabilizer of any given point. Identify this S^1 with complex numbers of absolute value 1. This gives an invariant almost complex structure on G/K. That is, each tangent space is a complex vector space. Almost complex structures don't always come from complex structures, but this one does (it is integrable). Notice that it is a little unexpected that G/K has a complex structure (G and K are odd dimensional in the case of $G = E_7$, $K = E_6 \oplus \mathbb{R}$, so they have no hope of having a complex structure).

Example 29.3. Let's look at E_6, with affine Dynkin diagram



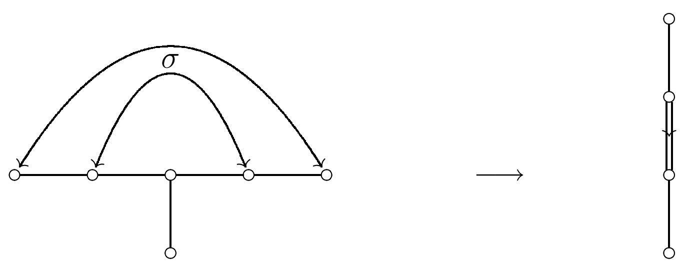
We get the possibilities

$\sum n_i m_i = 2$	# of ways	how to do it	maximal compact subgroup K
2×1	one way		E_6 (compact form)
1×2	one way		$A_1 A_5$
$1 \times 1 + 1 \times 1$	one way		$D_5 \oplus \mathbb{R}$

In the last one, the maximal compact subalgebra is $D_5 \oplus \mathbb{R}$. Just as before, we get a Hermitian symmetric space. Let's compute its dimension (over $\mathbb{C}$). The dimension will be the dimension of E_6 minus the dimension of $D_5 \oplus \mathbb{R}$, all divided by 2 (because we want complex dimension), which is $(78 - 46)/2 = 16$.

So we have found two non-compact simply connected Hermitian symmetric spaces of dimensions 16 and 27. These are the only "exceptional" cases; all the others fall into infinite families!

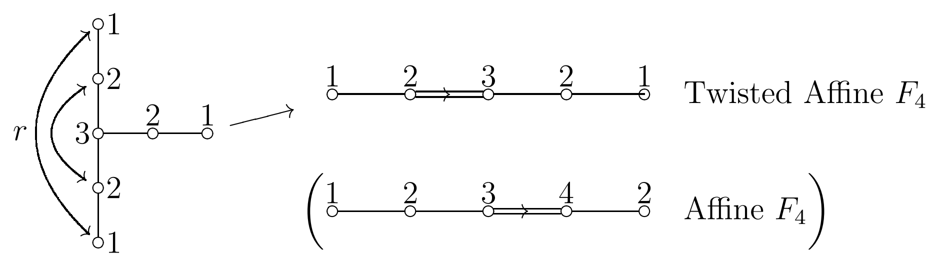
There are also some OUTER automorphisms of E_6 coming from the diagram automorphism



The fixed point subalgebra has Dynkin diagram obtained by folding the E_6 on itself. This is the F_4 Dynkin diagram. The fixed points of E_6 under the diagram automorphism is an F_4 Lie algebra. So we get a real form of E_6 with maximal compact subgroup F_4. This is probably the easiest way to construct F_4, by the way. Moreover, we can decompose E_6 as a representation of F_4. $\dim E_6 = 78$ and $\dim F_4 = 52$, so $E_6 = F_4 \oplus 26$, where 26 turns out to be irreducible (the smallest non-trivial representation of F_4 ... the only one anybody actually works with). The roots of F_4 look like $(\dots, \pm 1, \pm 1 \dots)$ (24 of these) and $(\pm\frac{1}{2} \cdots \pm \frac{1}{2})$ (16 of these), and $(\dots, \pm 1 \dots)$ (8 of them) ... the last two types are in the same orbit of the Weyl group.

The 26 dimensional representation has the following character: it has all norm 1 roots with multiplicity 1 and 0 with multiplicity 2 (note that this is not minuscule).

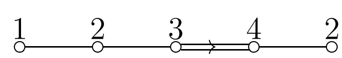
There is one other real form of E_6. To get at it, we have to talk about Kac's description of non-inner automorphisms of order N. The non-inner automorphisms all turn out to be related to diagram automorphisms. Choose a diagram automorphism of order r, which divides N. Let's take the standard thing on E_6. Fold the diagram (take the fixed points), and form a TWISTED affine Dynkin diagram (note that the arrow goes the wrong way from the affine F_4)



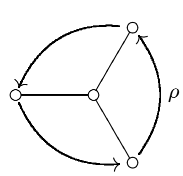
There are also numbers on the twisted diagram, but nevermind them. Find n_i so that $r\sum n_i m_i = N$. This is Kac's general rule. We'll only use the case $N = 2$.

If $r > 1$, the only possibility is $r = 2$ and one n_1 is 1 and the corresponding m_i is 1. So we just have to find points of weight 1 in the twisted affine Dynkin diagram. There are just two ways of doing this in the case of E_6

one of these gives us F_4, and the other has maximal compact subalgebra C_4, which is the split form since $\dim C_4 = \#\text{roots of } F_4/2 = 24$.

Example 29.4. F_4. The affine Dynkin is  We can cross out one node of weight 1, giving the compact form (split form), or a node of weight 2 (in two ways), giving maximal compacts A_1C_3 or B_4. This gives us three real forms.

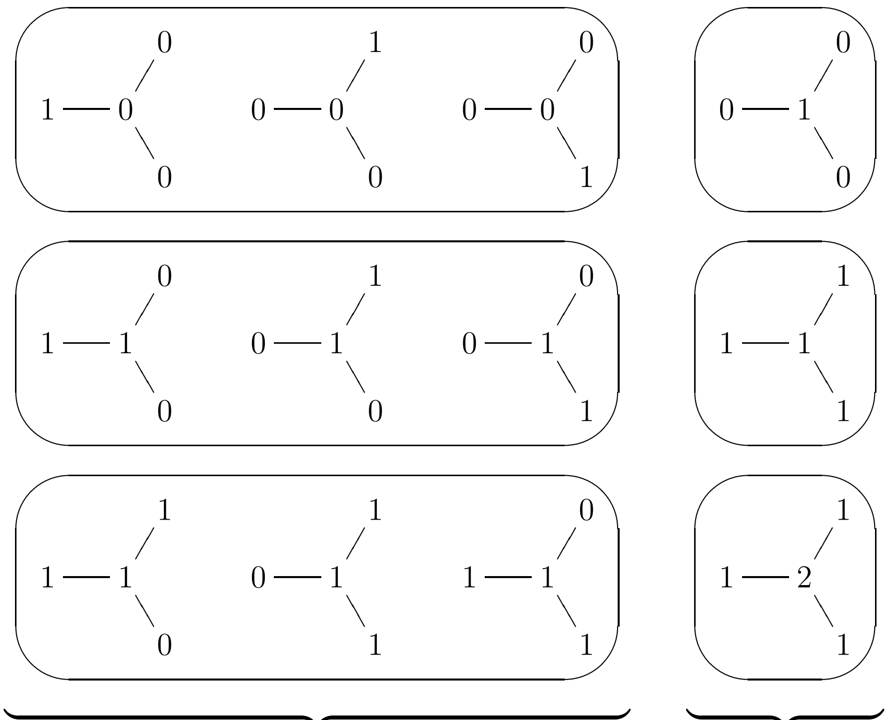
Example 29.5. G_2. We can actually draw this root system ... UCB won't supply me with a four dimensional board. The construction is to take the D_4 algebra and look at the fixed points of:



We want to find the fixed point subalgebra.

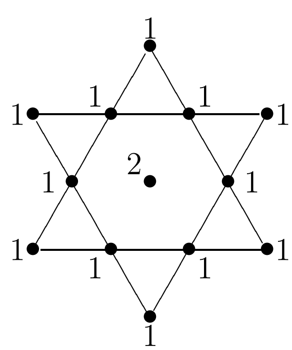
Fixed points on Cartan subalgebra: ρ fixes a two dimensional space, and has 1 dimensional eigenspaces corresponding to ω and $\bar{\omega}$, where $\omega^3 = 1$. The 2 dimensional space will be the Cartan subalgebra of G_2.

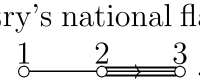
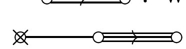
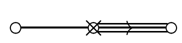
Positive roots of D_4 as linear combinations of simple roots (not fundamental weights):



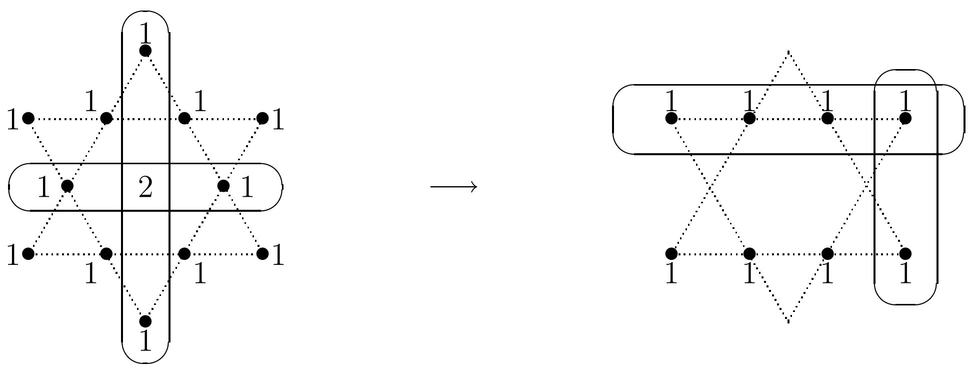
projections of norm $2/3$ projections of norm 2

There are six orbits under ρ, grouped above. It obviously acts on the negative roots in exactly the same way. What we have is a root system with six roots of norm 2 and six roots of norm $2/3$. Thus, the root system is G_2:



One of the only root systems to appear on a country's national flag. Now let's work out the real forms. Look at the affine: . we can delete the node of weight 1, giving the compact form: . We can delete the node of weight 2, giving $A_1 A_1$ as the compact subalgebra:  ... this must be the split form because there is nothing else the split form can be.

Let's say some more about the split form. What does the Lie algebra of G_2 look like as a representation of the maximal compact subalgebra $A_1 \times A_1$? In this case, it is small enough that we can just draw a picture:



We have two orthogonal A_1s, and we have leftover the stuff on the right. This thing on the right is a tensor product of the 4 dimensional irreducible representation of the horizontal and the 2 dimensional of the vertical. Thus, $G_2 = 3 \times 1 + 1 \otimes 3 + 4 \otimes 2$ as irreducible representations of $A_1^{(\text{horizontal})} \otimes A_1^{(\text{vertical})}$.

Let's use this to determine exactly what the maximal compact subgroup is. It is a quotient of the simply connected compact group $SU(2) \times SU(2)$, with Lie algebra $A_1 \times A_1$. Just as for E_8, we need to identify which elements of the center act trivially on G_2. The center is $\mathbb{Z}/2 \times \mathbb{Z}/2$. Since we've decomposed G_2, we can compute this easily. A non-trivial element of the center of $SU(2)$ acts as 1 (on odd dimensional representations) or -1 (on even dimensional representations). So the element $z \times z \in SU(2) \times SU(2)$ acts trivially on $3 \otimes 1 + 1 \otimes 3 + 4 \times 2$. Thus the maximal compact subgroup of the non-compact simple G_2 is $SU(2) \times SU(2)/(z \times z) \cong SO_4(\mathbb{R})$, where z is the non-trivial element of $\mathbb{Z}/2$.

So we have constructed $3 + 4 + 5 + 3 + 2$ (from E_8, E_7, E_6, F_4, G_2) real forms of exceptional simple Lie groups.

There are another 5 exceptional real Lie groups: Take COMPLEX groups $E_8(\mathbb{C})$, $E_7(\mathbb{C})$, $E_6(\mathbb{C})$, $F_4(\mathbb{C})$, and $G_2(\mathbb{C})$, and consider them as REAL. These give simple real Lie groups of dimensions 248×2, 133×2, 78×2, 52×2, and 14×2.

Lecture 30 - Irreducible unitary representations of $SL_2(\mathbb{R})$

$SL_2(\mathbb{R})$ is non-compact. For compact Lie groups, all unitary representations are finite dimensional, and are all known well. For non-compact groups, the theory is much more complicated. Before doing the infinite dimensional representations, we'll review finite dimensional (usually not unitary) representations of $SL_2(\mathbb{R})$.

Finite dimensional representations

Finite dimensional complex representations of the following are much the same: $SL_2(\mathbb{R})$, $\mathfrak{sl}_2\mathbb{R}$, $\mathfrak{sl}_2\mathbb{C}$ [branch $SL_2(\mathbb{C})$ as a complex Lie group] (as a complex Lie algebra), $\mathfrak{su}_2\mathbb{R}$ (as a real Lie algebra), and SU_2 (as a real Lie group). This is because finite dimensional representations of a simply connected Lie group are in bijection with representations of the Lie algebra. Complex representations of a REAL Lie algebra L correspond to complex representations of its complexification $L \otimes \mathbb{C}$ considered as a COMPLEX Lie algebra.

Note: Representations of a COMPLEX Lie algebra $L \otimes \mathbb{C}$ are not the same as representations of the REAL Lie algebra $L \otimes \mathbb{C} \cong L + L$. The representations of the real Lie algebra correspond roughly to (reps of L)$\otimes$(reps of L).

Strictly speaking, $SL_2(\mathbb{R})$ is not simply connected, which is not important for finite dimensional representations.

Recall the main results for representations of SU_2:

1. For each positive integer n, there is one irreducible representation of dimension n.

2. The representations are completely reducible (every representation is a sum of irreducible ones). This is perhaps the most important fact.

   The finite dimensional representation theory of SU_2 is EASIER than the representation theory of the ABELIAN Lie group $\mathbb{R}^2$, and that is because representations of SU_2 are completely reducible.

   For example, it is very difficult to classify pairs of commuting nilpotent matrices.

Completely reducible representations:

1. Complex representations of finite groups.

2. Representations of compact groups (Weyl character formula)

3. More generally, unitary representations of anything (you can take orthogonal complements of subrepresentations)

4. Finite dimensional representations of semisimple Lie groups.

Representations which are not completely reducible:

1. Representations of a finite group G over fields of characteristic $p| \, |G|$.

2. Infinite dimensional representations of non-compact Lie groups (even if they are semisimple).

We'll work with the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, which has basis $H = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. H is a basis for the Cartan subalgebra $\left(\begin{smallmatrix}a & 0\\ 0 & -a\end{smallmatrix}\right)$. E spans the root space of the simple root. F spans the root space of the negative of the simple root. We find that $[H, E] = 2E$, $[H, F] = -2F$ (so E and F are eigenvectors of H), and you can check that $[E, F] = H$.

$$\begin{array}{cccl} -2 & 0 & 2 & \longleftarrow \text{weights} = \text{eigenvalues under } H \\ \bullet & \bullet & \bullet & \\ F & H & E & \end{array}$$

Weyl group of order 2

The Weyl group is generated by $\omega = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and $\omega^2 = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Let V be a finite dimensional irreducible complex representation of $\mathfrak{sl}_2\mathbb{R}$. First decompose V into eigenspaces of the Cartan subalgebra (weight spaces) (i.e. eigenspaces of the element H). Note that eigenspaces of H exist because V is FINITE-DIMENSIONAL (remember this is a complex representation). Look at the LARGEST eigenvalue of H (exists since V is finite dimensional), with eigenvector v. We have that $Hv = nv$ for some n. Compute

$$\begin{aligned} H(Ev) &= [H, E]v + E(Hv) \\ &= 2Ev + Env = (n+2)Ev \end{aligned}$$

So $Ev = 0$ (lest it be an eigenvector of H with higher eigenvalue). $[E, -]$ increases weights by 2 and $[F, -]$ decreases weights by 2, and $[H, -]$ fixes weights.

We have that E kills v, and H multiplies it by n. What does F do to v?

$$\begin{tikzcd}
 & nv & (n-2)Fv & (n-4)F^2v & (n-6)F^3v & \cdots \\
0 & v \arrow[l, bend left, "E"] \arrow[u, "H"] \arrow[r, bend left, "F"] & Fv \arrow[l, bend left, "E"', "\times n"] \arrow[u, "H"] \arrow[r, bend left, "F"] & F^2v \arrow[l, bend left, "E"', "\times(2n-2)"] \arrow[u, "H"] \arrow[r, bend left, "F"] & F^3v \arrow[l, bend left, "E"', "\times(3n-6)"] \arrow[u, "H"] & \cdots
\end{tikzcd}$$

What is $E(Fv)$? Well,

$$\begin{aligned} EFv &= FEv + [E,F]v \\ &= 0 + Hv = nv \end{aligned}$$

In general, we have

$$\begin{aligned} H(F^i v) &= (n-2i)F^i v \\ E(F^i v) &= (ni - i(i-1))F^{i-1}v \\ F(F^i v) &= F^{i+1}v \end{aligned}$$

So the vectors $F^i v$ span V because they span an invariant subspace. This gives us an infinite number of vectors in distinct eigenspaces of H, and V is finite dimensional. Thus, $F^k v = 0$ for some k. Suppose k is the SMALLEST integer such that $F^k v = 0$. Then

$$0 = E(F^k v) = (nk - k(k-1))\underbrace{EF^{k-1}v}_{\neq 0}$$

So $nk - k(k-1) = 0$, and $k \neq 0$, so $n - (k-1) = 0$, so $\boxed{k = n+1}$. So V has a basis consisting of $v, Fv, \ldots, F^n v$. The formulas become a little better if we use the basis $w_n = v, w_{n-2} = Fv, w_{n-4} = \frac{F^2 v}{2!}, \frac{F^3 v}{3!}, \ldots, \frac{F^n v}{n!}$.

$$\begin{tikzcd}
 & & & & & & & E \\
w_{-6} \arrow[r, bend left, "1"] & w_{-4} \arrow[l, bend left, "6"] \arrow[r, bend left, "2"] & w_{-2} \arrow[l, bend left, "5"] \arrow[r, bend left, "3"] & w_0 \arrow[l, bend left, "4"] \arrow[r, bend left, "4"] & w_2 \arrow[l, bend left, "3"] \arrow[r, bend left, "5"] & w_4 \arrow[l, bend left, "2"] \arrow[r, bend left, "6"] & w_6 \arrow[l, bend left, "1"] & \\
 & & & & & & & F
\end{tikzcd}$$

This says that $E(w_2) = 5w_4$ for example. So we've found a complete description of all finite dimensional irreducible complex representations of $\mathfrak{sl}_2\mathbb{R}$. This is as explicit as you could possibly want.

These representations all lift to the group $SL_2(\mathbb{R})$: $SL_2(\mathbb{R})$ acts on homogeneous polynomials of degree n by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(x,y) = f(ax+by, cx+dy)$. This is an $n+1$ dimensional space, and you can check that the eigenspaces are $x^i y^{n-i}$.

We have implicitly constructed VERMA MODULES. We have a basis $w_n, w_{n-2}, \ldots, w_{n-2i}, \ldots$ with relations $H(w_{n-2i}) = (n-2i)w_{n-2i}$, $Ew_{n-2i} = (n-i+1)w_{n-2i+2}$, and $Fw_{n-2i} = (i+1)w_{n-2i-2}$. These are obtained by copying the formulas from the finite dimensional case, but allow it to be infinite dimensional. This is the universal representation generated by the highest weight vector w_n with eigenvalue n under H (highest weight just means $E(w_n) = 0$).

Let's look at some things that go wrong in infinite dimensions.

Warning 30.1. Representations corresponding to the Verma modules do NOT lift to representations of $SL_2(\mathbb{R})$, or even to its universal cover. The reason: look at the Weyl group (generated by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$) of $SL_2(\mathbb{R})$ acting on $\langle H\rangle$; it changes H to $-H$. It maps eigenspaces with eigenvalue m to eigenvalue $-m$. But if you look at the Verma module, it has eigenspaces $n, n-2, n-4, \ldots$, and this set is obviously not invariant under changing sign. The usual proof that representations of the Lie algebra lifts uses the exponential map of matrices, which doesn't converge in infinite dimensions.

Remark 30.2. The universal cover $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$, or even the double cover $Mp_2(\mathbb{R})$, has NO faithful finite dimensional representations.

Proof. Any finite dimensional representation comes from a finite dimensional representation of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$. All such finite dimensional representations factor through $SL_2(\mathbb{R})$. $\square$

All finite dimensional representations of $SL_2(\mathbb{R})$ are completely reducible. Weyl did this by Weyl's unitarian trick:

Notice that finite dimensional representations of $SL_2(\mathbb{R})$ are isomorphic (sort of) to finite dimensional representations of the COMPACT group SU_2 (because they have the same complexified Lie algebras. Thus, we just have to show it for SU_2. But representations of ANY compact group are completely reducible. Reason:

1. All unitary representations are completely reducible (if $U \subseteq V$, then $V = U \oplus U^\perp$).

2. Any representation V of a COMPACT group G can be made unitary: take any unitary form on V (not necessarily invariant under G), and average it over G to get an invariant unitary form. We can average because G is compact, so we can integrate any continuous function over G. This form is positive definite since it is the average of positive definite forms (if you try this with non-(positive definite) forms, you might get zero as a result).

The Casimir operator

Set $\Omega = 2EF + 2FE + H^2 \in U(\mathfrak{sl}_2\mathbb{R})$. The main point is that Ω commutes with $\mathfrak{sl}_2\mathbb{R}$. You can check this by brute force:

$$\begin{aligned}
[H,\Omega] &= 2\underbrace{\left([H,E]F + E[H,F]\right)}_{0} + \cdots \\
[E,\Omega] &= 2[E,E]F + 2E[F,E] + 2[E,F]E \\
&\qquad + 2F[E,E] + [E,H]H + H[E,H] = 0 \\
[F,\Omega] &= \text{Similar}
\end{aligned}$$

Thus, Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$. In fact, it generates the center. This doesn't really explain where Ω comes from.

Remark 30.3. Why does Ω exist? The answer is that it comes from a symmetric invariant bilinear form on the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ given by $(E,F)=1$, $(E,E)=(F,F)=(F,H)=(E,H)=0$, $(H,H)=2$. This bilinear form is an invariant map $L\otimes L\to\mathbb{C}$, where $L=\mathfrak{sl}_2\mathbb{R}$, which by duality gives an invariant element in $L\otimes L$, which turns out to be $2E\otimes F+2F\otimes E+H\otimes H$. The invariance of this element corresponds to Ω being in the center of $U(\mathfrak{sl}_2\mathbb{R})$.

Since Ω is in the center of $U(\mathfrak{sl}_2\mathbb{R})$, it acts on each irreducible representation as multiplication by a constant. We can work out what this constant is for the finite dimensional representations. Apply Ω to the highest vector w_n:

$$\begin{aligned}(2EF+2FE+HH)w_n &= (2n+0+n^2)w_n\\ &= (2n+n^2)w_n\end{aligned}$$

So Ω has eigenvalue $2n+n^2$ on the irreducible representation of dimension $n+1$. Thus, Ω has DISTINCT eigenvalues on different irreducible representations, so it can be used to separate different irreducible representations. The main use of Ω will be in the next lecture, where we'll use it to deal with infinite dimensional representation.

To finish today's lecture, let's look at an application of Ω. We'll sketch an algebraic argument that the representations of $\mathfrak{sl}_2\mathbb{R}$ are completely reducible. Given an exact sequence of representations

$$0\to U\to V\to W\to 0$$

we want to find a splitting $W\to V$, so that $V=U\oplus W$.

Step 1: Reduce to the case where $W=\mathbb{C}$. The idea is to look at

$$0\to \mathrm{Hom}_{\mathbb{C}}(W,U)\to \mathrm{Hom}_{\mathbb{C}}(W,V)\to \mathrm{Hom}_{\mathbb{C}}(W,W)\to 0$$

and $\mathrm{Hom}_{\mathbb{C}}(W,W)$ has an obvious one dimensional subspace, so we can get a smaller exact sequence

$$0\to \mathrm{Hom}_{\mathbb{C}}(W,U)\to \text{subspace of } \mathrm{Hom}_{\mathbb{C}}(W,V)\to \mathbb{C}\to 0$$

and if we can split this, the original sequence splits.

Step 2: Reduce to the case where U is irreducible. This is an easy induction on the number of irreducible components of U.

► **Exercise 30.1.** Do this.

Step 3: This is the key step. We have

$$0 \to U \to V \to \mathbb{C} \to 0$$

with U irreducible. Now apply the Casimir operator Ω. V splits as eigenvalues of Ω, so is $U \oplus \mathbb{C}$ UNLESS U has the same eigenvalue as $\mathbb{C}$ (i.e. unless $U = \mathbb{C}$).

Step 4: We have reduced to

$$0 \to \mathbb{C} \to V \to \mathbb{C} \to 0$$

which splits because $\mathfrak{sl}_2(\mathbb{R})$ is perfect[1] (no homomorphisms to the abelian algebra $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$).

Next time, in the final lecture, we'll talk about infinite dimensional unitary representations.

[1] L is *perfect* if $[L, L] = L$

Lecture 31 - Unitary representations of $SL_2(\mathbb{R})$

Last lecture, we found the finite dimensional (non-unitary) representations of $SL_2(\mathbb{R})$.

Background about infinite dimensional representations

(of a Lie group G) What is an finite dimensional representation?

1st guess Banach space acted on by G?

This is no good for some reasons: Look at the action of G on the functions on G (by left translation). We could use L^2 functions, or L^1 or L^p. These are completely different Banach spaces, but they are essentially the same representation.

2nd guess Hilbert space acted on by G? This is sort of okay.

The problem is that finite dimensional representations of $SL_2(\mathbb{R})$ are NOT Hilbert space representations, so we are throwing away some interesting representations.

Solution (Harish-Chandra) Take $\mathfrak{g}$ to be the Lie algebra of G, and let K be the maximal compact subgroup. If V is an infinite dimensional representation of G, there is no reason why $\mathfrak{g}$ should act on V.

The simplest example fails. Let $\mathbb{R}$ act on $L^2(\mathbb{R})$ by left translation. Then the Lie algebra is generated by $\frac{d}{dx}$ (or $i\frac{d}{dx}$) acting on $L^2(\mathbb{R})$, but $\frac{d}{dx}$ of an L^2 function is not in L^2 in general.

Let V be a Hilbert space. Set V_ω to be the K-finite vectors of V, which are the vectors contained in a finite dimensional representation of K. The point is that K is compact, so V splits into a Hilbert space direct sum finite dimensional representations of K, at least if V is a Hilbert space. Then V_ω is a representation of the Lie algebra $\mathfrak{g}$, not a representation of G. V_ω is a representation of the group K. It is a $(\mathfrak{g}, K)$-module, which means that it is acted on by $\mathfrak{g}$ and K in a "compatible" way, where compatible means that

1. they give the same representations of the Lie algebra of K.
2. $k(u)v = k(u(k^{-1}v))$ for $k \in K$, $u \in \mathfrak{g}$, and $v \in V$.

The K-finite vectors of an irreducible unitary representation of G is ADMISSIBLE, which means that every representation of K only occurs a *finite* number of times. The GOOD category of representations is the representations of admissible $(\mathfrak{g}, K)$-modules. It turns out that this is a really well behaved category.

We want to find the unitary irreducible representations of G. We will do this in several steps:

1. Classify all irreducible admissible representations of G. This was solved by Langlands, Harish-Chandra et. al.

2. Find which have hermitian inner products $(\ , \ )$. This is easy.

3. Find which ones are positive definite. This is VERY HARD. We'll only do this for the simplest case: $SL_2(\mathbb{R})$.

The group $SL_2(\mathbb{R})$

We found some generators (in $Lie(SL_2(\mathbb{R})) \otimes \mathbb{C}$ last time: E, F, H, with $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. We have that $H = -i\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $E = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, and $F = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$. Why not use the old $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$?

Because $SL_2(\mathbb{R})$ has two different classes of Cartan subgroup: $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the second one is COMPACT. The point is that non-compact (abelian) groups need not have eigenvectors on infinite dimensional spaces. An eigenvector is the same as a weight space. The first thing you do is split it into weight spaces, and if your Cartan subgroup is not compact, you can't get started. We work with the compact subalgebra so that the weight spaces exist.

Given the representation V, we can write it as some direct sum of eigenspaces of H, as the Lie group H generates is compact (isomorphic to S^1). In the finite dimensional case, we found a HIGHEST weight, which gave us complete control over the representation. The trouble is that in infinite dimensions, there is no reason for the highest weight to exist, and in general they don't. The highest weight requires a finite number of eigenvalues.

A good substituted for the highest weight vector: Look at the Casimir operator $\Omega = 2EF + 2FE + H^2 + 1$. The key point is that Ω is in the center of the universal enveloping algebra. As V is assumed admissible, we can conclude that Ω has eigenvectors (because we can find a finite dimensional space acted on by Ω). As V is irreducible and Ω commutes with G, all of V is an eigenspace of Ω. We'll see that this gives us about as much information as a highest weight vector.

Let the eigenvalue of Ω on V be λ^2 (the square will make the interesting representations have integral λ; the $+1$ in Ω is for the same reason).

Suppose $v \in V_n$, where V_n is the space of vectors where H has eigenvalue n. In the finite dimensional case, we looked at Ev, and saw that $HEv = (n+2)Ev$. What is FEv? If v was a highest weight vector, we could control this. Notice that $\Omega = 4FE + H^2 + 2H + 1$ (using $[E, F] = H$), and $\Omega v = \lambda^2 v$. This says that $4FEv + n^2v + 2nv + v = \lambda^2 v$. This shows that FEv is a multiple of v.

Now we can draw a picture of what the representation looks like:

$$\begin{tikzcd}
\cdots \arrow[r, bend left, dotted] & v_{n-4} \arrow[l, bend left] \arrow[r, bend left, dotted] & v_{n-2} \arrow[l, bend left] \arrow[r, bend left, dotted] & v_n \arrow[l, bend left] \arrow[r, bend left] & v_{n+2} \arrow[l, bend left, "\left(\frac{n^2+2n+1-\lambda^2}{4}\right)"] \arrow[r, bend left] & v_{n+4} \arrow[l, bend left, dotted] \arrow[r, bend left] & \cdots \arrow[l, bend left, dotted] & \begin{matrix} E \\ H \\ F \end{matrix}
\end{tikzcd}$$

Thus, V_ω is spanned by V_{n+2k}, where k is an integer. The non-zero elements among the V_{n+2k} are linearly independent as they have different eigenvalues. The only question remaining is whether any of the V_{n+2k} vanish.

There are four possible shapes for an irreducible representation

– infinite in both directions: $\begin{tikzcd} \cdots \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd} \begin{matrix} E \\ H \\ F \end{matrix}$

– a lowest weight, and infinite in the other direction:

$$\begin{tikzcd} \cdots \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} \quad \begin{matrix} E \\ H \\ F \end{matrix}$$

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– we have a highest weight and a lowest weight, in which case it is finite dimensional $\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} \quad \begin{matrix} E \\ H \\ F \end{matrix}$

We'll see that all these show up. We also see that an irreducible representation is completely determined once we know λ and some n for which $V_n \neq 0$. The remaining question is to construct representations with all possible values of $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$. n is an integer because it must be a representations of the circle.

If n is even, we have

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "\frac{\lambda-7}{2}"] & -6 \arrow[l, bend left, "\frac{\lambda+7}{2}"] \arrow[r, bend left, "\frac{\lambda-5}{2}"] & -4 \arrow[l, bend left, "\frac{\lambda+5}{2}"] \arrow[r, bend left, "\frac{\lambda-3}{2}"] & -2 \arrow[l, bend left, "\frac{\lambda+3}{2}"] \arrow[r, bend left, "\frac{\lambda-1}{2}"] & 0 \arrow[l, bend left, "\frac{\lambda+1}{2}"] \arrow[r, bend left, "\frac{\lambda+1}{2}"] & 2 \arrow[l, bend left, "\frac{\lambda-1}{2}"] \arrow[r, bend left, "\frac{\lambda+3}{2}"] & 4 \arrow[l, bend left, "\frac{\lambda-3}{2}"] \arrow[r, bend left, "\frac{\lambda+5}{2}"] & 6 \arrow[l, bend left, "\frac{\lambda-5}{2}"] \arrow[r, bend left, "\frac{\lambda+7}{2}"] & \cdots \arrow[l, bend left, "\frac{\lambda-7}{2}"] & \begin{matrix} E \\ H \\ F \end{matrix}
\end{tikzcd}$$

It is easy to check that these maps satisfy $[E,F]=H$, $[H,E]=2E$, and $[H,F]=-2F$

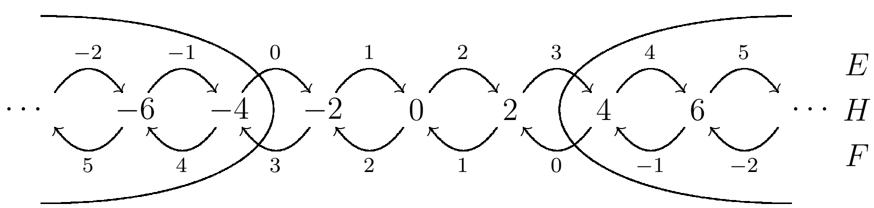
▶ **Exercise 31.1.** Do the case of n odd.

Problem: These may not be irreducible, and we want to decompose them into irreducible representations. The only way they can fail to be irreducible if if $Ev_n = 0$ of $Fv_n = 0$ for some n (otherwise, from any vector, you can generate the whole space). The only ways that can happen is if

n even: λ an odd integer
n odd: λ an even integer.

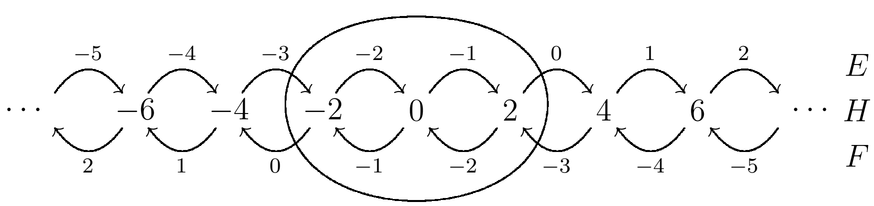
What happens in these cases? The easiest thing is probably just to write out an example.

Example 31.1. Take n even, and $\lambda = 3$, so we have



You can just see what the irreducible subrepresentations are ... they are shown in the picture. So V has two irreducible subrepresentations V_- and V_+, and $V/(V_- \oplus V_+)$ is an irreducible 3 dimensional representation.

Example 31.2. If n is even, but λ is negative, say $\lambda = -3$, we get



Here we have an irreducible finite dimensional representation. If you quotient out by that subrepresentation, you get $V_+ \oplus V_-$.

▶ **Exercise 31.2.** Show that for n odd, and $\lambda = 0$, $V = V_+ \oplus V_-$.

So we have a complete list of all irreducible admissible representations:

1. if $\lambda \notin \mathbb{Z}$, you get one representation (remember $\lambda \equiv -\lambda$). This is the bi-infinite case.

2. Finite dimensional representation for each $n \geq 1$ ($\lambda = \pm n$)

3. Discrete series for each $\lambda \in \mathbb{Z} \smallsetminus \{0\}$, which is the half infinite case: you get a lowest weight when $\lambda < 0$ and a highest weight when $\lambda > 0$.

4. two "limits of discrete series" where n is odd and $\lambda = 0$.

Which of these can be made into *unitary* representations? $H^\dagger = -H$, $E^\dagger = F$, and $F^\dagger = E$. If we have a hermitian inner product $(\, , \,)$, we see that

$$\begin{aligned}(v_{j+2}, v_{j+2}) &= \frac{2}{\lambda + j + 1}(Ev_j, v_{j+2}) \\ &= \frac{2}{\lambda + j + 1}(v_j, -Fv_{j+2}) \\ &= -\frac{2}{\lambda + j + 1}\frac{\overline{\lambda - j - 1}}{2}(v_j, v_j) > 0\end{aligned}$$

where we fix the sign errors. So we want $-\frac{\overline{\lambda-1-j}}{\lambda+j+1}$ to be real and positive whenever $j, j+2$ are non-zero eigenvectors. So

$$-(\lambda - 1 - j)(\lambda + 1 + j) = -\lambda^2 + (j+1)^2$$

should be positive for all j. Conversely, when you have this, blah.

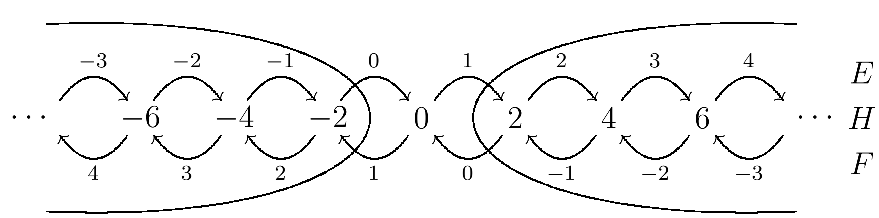
This condition is satisfied in the following cases:

1. $\lambda^2 \leq 0$. These representations are called PRINCIPAL SERIES representations. These are all irreducible *except* when $\lambda = 0$ and n is odd, in which case it is the sum of two limits of discrete series representations

2. $0 < \lambda < 1$ and j even. These are called COMPLEMENTARY SERIES. They are annoying, and you spend a lot of time trying to show that they don't occur.

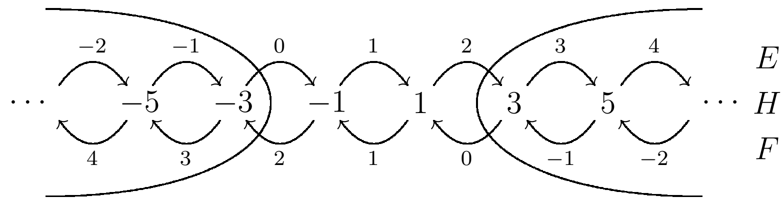
3. $\lambda^2 = n^2$ for $n \geq 1$ (for some of the irreducible pieces).

   If $\lambda = 1$, we get



   We see that we get two discrete series and a 1 dimensional representation, all of which are unitary

For $\lambda = 2$ (this is the more generic one), we have



The middle representation (where $(j+1)^2 < \lambda^2 = 4$ is NOT unitary, which we already knew. So the DISCRETE SERIES representations ARE unitary, and the FINITE dimensional representations of dimension greater than or equal to 2 are NOT.

Summary: the irreducible unitary representations of $SL_2(\mathbb{R})$ are given by

1. the 1 dimensional representation
2. Discrete series representations for any $\lambda \in \mathbb{Z} \setminus \{0\}$
3. Two limit of discrete series representations for $\lambda = 0$
4. Two series of principal series representations:

$$\begin{array}{l} j \text{ even: } \lambda \in i\mathbb{R},\ \lambda \geq 0 \\ j \text{ odd: } \lambda \in i\mathbb{R},\ \lambda > 0 \end{array}$$

5. Complementary series: parameterized by λ, with $0 < \lambda < 1$.

The nice stuff that happened for $SL_2(\mathbb{R})$ breaks down for more complicated Lie groups.

Representations of finite covers of $SL_2(\mathbb{R})$ are similar, except j need not be integral. For example, for the double cover $\widehat{SL_2(\mathbb{R})} = Mp_2(\mathbb{R})$, $2j \in \mathbb{Z}$.

▶ **Exercise 31.3.** Find the irreducible unitary representations of $Mp_2(\mathbb{R})$.