$II_{1,25}$
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Week 4, 29 Sep 2003, $II_{1,25}$

We review the even unimodular lattices, described in my first lecture. The basic type is $II_{m,n} = \{x_1, \ldots, x_{m+n} \in \mathbb{R}^{m+n} | (\text{all } x_i \in \mathbb{Z}$ **or** all $x_i \in \mathbb{Z} + \frac{1}{2})$ **and** $\sum x_i$ even$\}$ with quadratic form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$. This is even if and only if $m \equiv n \bmod 8$.

Any **indefinite** even unimodular lattice is isomorphic to some $II_{m,n}$. The story for definite lattices is somewhat different:

dimension	0	8	16	24	32
isom. types	1	1	2	24	$> 10^9$

The 8-dimensional lattice is $E_8 \cong II_{8,0}$, the two 16-dimensional lattices are $E_8 \oplus E_8$ and $II_{16,0}$, and the 24-dimensional lattices were classified by Niemeier. They will be described later on. As you can see, there is a sort of "phase change" at 24 dimensions.

There is a basic philosophy that is found in books on number theory and lattices, that all lattices (over all number fields) are equal. The philosophy of today is that some lattices are more equal than others. In particular, $II_{1,25}$ is very special.

The dimension 26 is related to:

1. Critical dimension in bosonic string theory

$25 - 1 = 24$ is related to:

1. $\Delta(\tau) = \eta(\tau)^{24}$, where Δ is the discriminant cusp form of weight 12, and η is the Dedekind function.
2. The constant term of $j(\tau) - 720 = q^{-1} + 24 + 196884q + \ldots$, which is the "right normalization" of this modular function.
3. The Eisenstein series $E_2 = 1 - 24\sum \sigma_1(n) q^n$.
4. Dimension of the Leech Lattice.
5. Sporadic group M_{24}
6. If we take a double cover $Mp_2(\mathbb{Z})$ of $SL_2(\mathbb{Z})$, we have $(Mp_2(\mathbb{Z}))^{ab} \cong \mathbb{Z}/24$.

First important property (found by Conway): The Dynkin diagram of $II_{1,25}$ is the Leech lattice Λ.

This may sound a bit strange to you. It's like the apocryphal story where you add apples and oranges, and get the answer in bananas. It will take some time to explain what this means.

We can model hyperbolic space as one component of the norm 1 vectors of $\mathbb{R}^{1,25}$, which make up a two-sheeted hyperboloid. The induced metric is uniformly hyperbolic. The automorphisms of $II_{1,25}$ form a discrete subgroup of $O_{1,25}(\mathbb{R})$, and they include reflections in the norm -2 vectors r (i.e. in the hyperplanes $r^{\perp}$):

$$v \mapsto v + (v,r)r$$

These generate a discrete reflection group on hyperbolic space.

Fix one region bounded by reflection hyperplanes. This is a fundamental domain for the reflection group. [Marty: Is this compact, or at least finite volume?] It is not compact - it has 24 orbits of cusps (but this isn't the same 24 as I mentioned earlier). It has infinite volume. If you think the only interesting fundamental domains are compact or finite volume, you're just going to have to change your mind.

Look at the walls of a fundamental domain. The set of walls is (more or less) the Dynkin diagram of the reflection group. More precisely, the Dynkin diagram has 1 vertex for each wall, and two vertices are joined by:

1. 0 lines if the walls meet at an angle of $\pi/2$.
2. 1 line if the walls meet at an angle of $\pi/3$.
3. A thick line if the walls meet at ∞ (this means the hyperplanes in Lorentz space intersect at a nontrivial norm 0 vector).
4. Dotted line if they don't meet, some other stuff if something else happens, although there aren't any fixed conventions for these last two.

Now, walls can be identified with norm -2 vectors (up to sign), and the norm -2 vectors corresponding to the walls of our fundamental domain (where we choose those with positive inner product with a fixed timelike vector), are called **simple roots**. Simple roots correspond bijectively with nodes of the Dynkin diagram.

Q: What are the simple roots of $II_{1,25}$?

A: Using the original coordinates, it is a complete mess. We will choose an alternative coordinate system.

Let Λ be the Leech lattice, and $\Lambda(-1)$ the lattice with norms multiplied by -1. Then $\Lambda(-1) \oplus II_{1,1}$ is even and unimodular, so it is isomorphic to $II_{1,25}$. [Someone (Noah?): Why we can't just switch signature at the outset? Borcherds: It leads to problems later on. This is one of the sign errors built into the structure of the universe that we can't do anything about.] We write vectors of $II_{1,25}$ as (λ, m, n), where $\lambda \in \Lambda$, $m, n \in \mathbb{Z}$. The norm of (λ, m, n) is $\lambda^2 - 2mn$. Here, we are using a different inner product on $II_{1,1}$ than the one defined earlier. There is a simple diagonal rotation which yields $II_{1,1} \cong \mathbb{Z}^2$ with $((a,b),(c,d)) = ad + bc$.

We define ρ to be $(0,0,1)$. This is a Weyl vector, with norm $\rho^2 = 0$. We have $\rho^\perp / < \rho > \cong \Lambda(-1)$. Conway showed that the simple roots are the vectors $r \in II_{1,25}$ such that:

1. $(r,r) = -2$
2. $(r,\rho) = 1$

The list of all such vectors is given by $\{(\lambda, 1, \frac{\lambda^2}{2} - 1) | \lambda \in \Lambda\}$. The 1 is necessary for the second condition, and the $\frac{\lambda^2}{2} - 1$ is necessary for the first. This implies the simple roots correspond bijectively to elements of Λ. It is easy to show that all of these roots are simple. It is hard to show that there are no other simple roots.

Sketch of Conway's Proof: Suppose (v, m, n) is some other simple root (so $m > 1$). Then the norm condition implies $-v^2 + 2mn = -2$. Since (v, m, n) has non-negative inner product with all other simple roots (note that the sign is reversed from the one for simple roots of positive definite lattices), $A := -(v, \lambda) + m(\frac{\lambda^2}{2} - 1) + n \geq 0$ for all λ. This implies $(\frac{v}{m} - \lambda)^2 = 2 + \frac{2}{m^2} + \frac{A}{m} > 2$. Conway, Parker, and Sloane proved that the Leech lattice has covering radius $\sqrt{2}$, using about 50 pages of rather tiresome calculations. This means $\mathbb{R}^{24}$ is covered by closed balls of radius $\sqrt{2}$ centered at the lattice points of Λ. Our calculations showed that $\frac{v}{m}$ has distance greater than $\sqrt{2}$ from all $\lambda \in \Lambda$, which is not possible.

With this result, we can calculate the full automorphism group of $II_{1,25}$. Unfortunately, we are out of time, so the remaining half of this one hour lecture will happen next week.

Week 5, 6 Oct 2003, $II_{1,25}$ (continued)

We start by recalling some notation. $II_{1,25}$ is the even 26-dimensional even unimodular lattice. Its reflection group is generated by reflections in norm -2 vectors (one can consider norm 2 vectors, but it is much harder to deal with, and doesn't give a hyperbolic reflection group). There is a Weyl vector ρ, such that r is a simple root if and only if $r^2 = -2$ and $(r, \rho) = 1$. We write $II_{1,25} \cong \Lambda \oplus II_{1,1}$, and elements (λ, m, n) have norm $-\lambda^2 + 2mn$. The simple roots are $r_\lambda = (\lambda, 1, \frac{\lambda^2}{2} - 1)$ for $\lambda \in \Lambda$. This depends on the fact that the covering radius of Λ is at most $\sqrt{2}$.

Applications: We analyze $\mathrm{Aut}(II_{1,25})$. It is isomorphic to $\{\pm 1\} \times \mathrm{Aut}^+(II_{1,25})$, where $\mathrm{Aut}^+(II_{1,25})$ is the autochronous group, whose elements map the positive timelike cone to itself. $\mathrm{Aut}^+(II_{1,25}) \cong$ (reflection group) $\cdot$ (Affine automorphisms of Λ), a semidirect product, and the group of affine automorphisms is $\Lambda \cdot \mathrm{Aut}(\Lambda)$, where the group on the left acts by translations, and the group on the right is a central extension of Conway's sporadic simple group Co_1 by $\mathbb{Z}/2$.

In general, Aut(lattice) = (reflection) $\cdot$ (Auts fixing some fundamental domain). Note that this only works for positive definite or lorentzian lattices, as other lattices don't have fundamental domains. To each fundamental domain, we have a Weyl vector ρ.

Q: What are the automorphisms of $II_{1,25}$ fixing $\rho = (0,0,1)$? This is equivalent to asking what automorphisms preserve $\rho^\perp/\langle\rho\rangle = \Lambda$.

A: Suppose $v \in \Lambda$. We get an automorphism taking $(\lambda, 1, n) \mapsto (\lambda + v, 1, *)$. Since automorphisms preserve norm, we get the third coordinate:

$$(\lambda, m, n) \mapsto (\lambda + mv, m, mv^2/2 + mn)$$

This gives an action of Λ on $II_{1,25}$, fixing $\rho = (0,0,1)$ and $\rho^\perp/\langle\rho\rangle$. One can check that these are all of the automorphisms with these properties.

Q: What does the fundamental domain of the reflection group of $II_{1,25}$ look like?

A: It is a subset of $\mathbb{H}^{25}$ (Hyperbolic space). [Konstantin: If the lattice sits in 26-dimensional space, shouldn't the fundamental domain be 26-dimensional? Borcherds: I'm just taking the norm 1 vectors, so we can apply our 25-dimensional hyperbolic intuition to see what's going on.] There are two models of hyperbolic space:

1. "upper-half space" $\mathbb{R}^+ \times \mathbb{R}^{24}$ (looks like $\mathbb{H}^2$. [some drawing of a line (actually 24-dimensional), with a bumpy curve over it. bumps touching the line are "cusps", bumps over the line are "corners", and the whole thing looks periodic, indicating translation-invariance under Λ. There is a "horrible point" at ∞, indicated by an upward arrow.] We get finite volume after modding out by the Λ action.

2. "open ball model" (like the unit disc in $\mathbb{C}$). [drawing of a disc, with bumpy curve on inside looking like the bumps above, but getting small toward a "messy point" somewhere on the boundary.]

Q: Can we classify cusps and corners?

A: Yes. There are 23 orbits of cusps under $\Lambda \cdot \mathrm{Aut}(\Lambda)$. These are very interesting - they correspond to the 23 orbits of "deep holes" of Λ, namely the corners of distance $\sqrt{2}$ from 0 in the Voronoi decomposition. There are 284 orbits of corners. They are rather boring. These correspond to the 284 orbits of of "shallow holes", which are corners of distance less than $\sqrt{2}$ from 0.

One slightly less boring example of a corner: Suppose we use the original coordinate system:

$$II_{1,25} = \{(x_0, \dots, x_{25}) \in \mathbb{R}^{1,25} | \text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \frac{1}{2}, \sum x_i \text{ even } \}$$

Here, our norm is $x_0^2 - x_1^2 - \cdots - x_{25}^2\}$. What is ρ in these coordinates? We want to take the simplest representative possible, so we set the first spacelike coordinate x_1 to be 0: (*,0,*,...,*). Now, this forces x_2 to be nonzero, since if it were zero, ρ would be perpendicular to the norm -2 vector $(0,1,-1,0,\ldots,0)$, and Λ has no norm 2 vectors. The simplest form it can take is then, $(*,0,1,*,\ldots,*)$. Now, x_3 cannot be $-1,0,1$ for the same reason, since otherwise it would be easy to find another norm -2 vector perpendicular to ρ. Continuing, we have $(*,0,1,2,\ldots,24)$, and in order to have norm zero, we need the first coordinate to satisfy $n^2 = 0^2 + 1^2 + 2^2 + \cdots + 24^2 = 70^2$. This is in fact (by a theorem of Watson) the only nontrivial ($m \geq 2$) solution to $0^2 + 1^2 + \cdots + m^2 = n^2$. As it happens, $\rho = (70,0,1,\ldots,24)$ does give $\rho^\perp/\langle\rho\rangle \cong \Lambda$, although this is not easy (due to Conway and Curtis). The vector $(1,0,\ldots,0)$ is a corner of the fundamental domain of the reflection group. The hyperplanes passing through this corner correspond to roots of the form $(0,\ldots,0,-1,1,0,\ldots,0)$, and $(0,1,1,0,\ldots,0)$. This gives the diagram D_{25}, where the last vector is one of the degree one vertices adjacent to the degree three vertex.

Other corners have somewhat less nice reflection groups associated to them.

The D_{25} lattice is $\{x_1,\ldots,x_{25} | x_i \in \mathbb{Z}, \sum x_i \text{ even}\}$. The roots are vectors with 2 entries ± 1 and 0 elsewhere: $(0,\ldots,\pm 1,0,\ldots,\pm 1,0,\ldots,0)$, and the simple roots are $(0,\ldots,0,-1,1,0,\ldots,0)$ and $(1,1,0,\ldots,0)$. The Weyl vector of this reflection group is $(0,1,2,\ldots,24)$.

Cusps: The Niemeier lattices other than Λ correspond to orbits of primitive norm 0 vectors of $II_{1,25}$ under its automorphism group, which correspond to orbits of cusps of the fundamental domain of the reflection group under $\Lambda \cdot \mathrm{Aut}(\Lambda)$, which correspond to orbits of deep holes of Λ. All of these sets have 23 elements. Actually, deep holes themselves correspond to cusps, since we are looking at orbits under the same group of automorphisms.

If L is a Niemeier lattice, $II_{1,25} \cong L \oplus II_{1,1}$. As with Λ, we can take coordinates (λ, m, n), set $w = (0,0,1)$ a norm zero Weyl vector, and we can retrieve L by taking $w^\perp/\langle w\rangle$.

Next week's seminar will be the remaining $3/4$ of today's seminar.

Week 6, 13 Oct 2003, $II_{1,25}$ (continued)

We have the following four classes:

1. Isomorphism classes of Niemeier lattices
2. Orbits of primitive norm 0 vectors in $II_{1,25}$ under $\mathrm{Aut}(II_{1,25})$
3. Orbits (under $\Lambda \cdot \mathrm{Aut}(\Lambda)$) of cusps of the fundamental domain of reflection
4. Orbits of $\{\text{deep holes}\} \cup \{\infty\}$ under $\Lambda \cdot \mathrm{Aut}(\Lambda)$

The correspondence between 1 and 2 works for any $II_{1,8n+1}$. The correspondence between 2 and 3 works for hyperbolic reflection groups in general, with $\Lambda \cdot \mathrm{Aut}(\Lambda)$ is replaced by Aut(fundamental domain).

Write $II_{1,25} = \Lambda(-1) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and write elements as (λ, m, n) with norm $-\lambda^2 + 2mn$. Let $\rho = (0,0,1)$ be the Weyl vector in a chosen fundamental domain, so $\rho^\perp/\rho \cong \Lambda$. Now suppose (v,m,n) is a norm 0 vector in $II_{1,25}$. Then $v/m \in \Lambda \otimes \mathbb{Q}$. If (v,m,n) is in the fundamental domain

$$\{((v,m,n),(\lambda,1,\lambda^2/2-1)) \geq 0 \forall \lambda \in \Lambda\}$$

then $(v/m - \lambda)^2 \geq 2$ for all λ (Exercise).

More generally, the map $(v,m,n) \mapsto v/m$ gives a (non-smooth) one-to-one correspondence between the boundary of the fundamental domain and $(\Lambda \otimes \mathbb{R}) \cup \infty$. [Draws a picture of a spiky blob inscribed in a circle]

Example: Let $L = E_8$. What does the corresponding cusp look like? We write $II_{1,25} = L \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and write elements as $(v,m,n), v \in E_8^3$ with norm $-v^2 + 2mn$. A norm 0 vector corresponding to L is $w = (0,0,1)$. What are the simple roots whose hyperplanes pass through w, and what are the angles between them?

Let (r,m,n) be a vector in $II_{1,25}$, orthogonal to w, and with norm -2. The orthogonality condition implies $m = 0$, so we have $(r,0,n)$. The norm condition implies $r^2 = 2$, so r is a root of E_8^3, and there are 720 of these.

The vectors $(r,0,n)$ then form the roots of the **affine** reflection group of E_8. Pick some fundamental domain of the affine reflection group, and write down the simplest set of simple roots for the group. We have $(r_i,0,0)$ for r_i a simple root of E_8^3 (24 choices), and $(r',0,1)$ for r' a root of E_8^3 in the fundamental domain, and sort of $(0,1,-1)$, whose hyperplane doesn't actually pass through the cusp, but comes close to it. [Draws three copies of affine E_8, and draws an extra vertex, connected to each of the three diagrams by the vertex furthest out from the branch.]

Consequences:

1. What is (ρ,w)? ρ is a norm 0 vector with $(\rho,r) = 1$ for all simple roots r, and $w = \sum m_i r_i$, where r_i run through all simple roots in some component of affine E_8. $(\rho,w) = \sum m_i(\rho,r_i) = \sum m_i$, which is the Coxeter number h of that component of the root system. For E_8, $h = 30$. This implies a result of Niemeier, namely that all components of the Dynkin diagram of a Niemeier lattice L have the same Coxeter number.

2. The rank of the root system is 24, if $L \neq \Lambda$. We omit the proof, although it follows easily from the fact that L corresponds to a deep hole.

We use these two properties to classify Niemeier lattices, by listing all admissible root systems. We will make a table of simply laced affine diagrams, and assign positive integers to the vertices such that the number on any vertex is half the sum of those on the adjacent vertices, and such that these numbers are smallest possible. The Coxeter number of the diagram is the sum of these numbers.

Diagram	Coxeter number h
A_n	$n+1$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30

Then we make a list of possible Coxeter numbers, and the diagrams which correspond to them.

2	3	4	5	6	7	8	9	10	11	12
A_1	A_2	A_3	A_4	A_5, D_4	A_6	A_7, D_5	A_8	A_9, D_6	A_{10}	A_{11}, D_7, E_6

13	14	15	16	17	18	19	20	21
A_{12}	A_{13}, D_8	A_{14}	A_{15}, D_9	A_{16}	A_{17}, D_{10}, E_7	A_{18}	A_{19}, D_{11}	A_{20}

22	23	24	25	...	30	...	46
A_{21}, D_{12}	A_{22}	A_{23}, D_{13}	A_{24}	...	A_{29}, D_{16}, E_8	...	A_{45}, D_{24}

Now, we pick diagrams from each column so that their subscripts add to 24: A_1^{24}, A_2^{12}, A_3^8, A_4^6, $A_5^4 D_4$, D_4^6, A_6^4, $A_7^2 D_5^2$, A_8^3, $A_9^2 D_6$, D_6^4, $A_{11} D_7 E_6$, E_6^4, A_{12}^2, D_8^3, $A_{15} D_9$, $A_{17} E_7$, $D_{10} E_7^2$, D_{12}^2, A_{24}, $D_{16} E_8$, E_8^3, D_{24}.

Amazing fact: Each of these Dynkin diagrams corresponds to **exactly one** Neimeier lattice. The only known proofs of this are through case-by-case analysis.

This gives rise to a "Leech lattice calculus" which is mostly due to Conway. Suppose we find an affine Dynkin diagram in Λ. Then it is contained in the Dynkin diagram of some Niemeier lattice. For example: [Some messy drawing involving three copies of affine E_8 with an extra vertex connecting them by the long ends.] Take an affine D_{16} here [circles a D_{16}]. It **must** be contained in an affine $D_{16}E_8$, and this must contain an **orthogonal** affine E_8, so we can deduce the existence of this vertex here [Draws another vertex, and connects it to the second vertex in the long path of the copy of E_8 not included in the D_{16}.] This must have inner product 2 with the norm 0 vector of E_8^3, so these edges must exist. [draws edges to the ends of the length 3 paths of the E_8's involved with the D_{16}. Since there is an affine E_6 here, we must have E_6^4. [I didn't understand this, either]

Next week, Noah Snyder will talk about the Leech lattice.

Week 8, 27 Oct 2003, Applications of $II_{1,25}$

Recall: Λ is the Dynkin diagram of $II_{1,25} = \Lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The fundamental domain of $II_{1,25}$ has something to do with Λ(affine), where walls correspond to points of Λ, and cusps (i.e. norm 0 vectors) correspond to deep holes (23 orbits), which correspond to Niemeier lattices other than Λ.

Next, we will see how Λ controls unimodular lattices in small dimensions (≤ 25).

Recall that the Smith-Minkowski-Siegel mass formula implies the number of unimodular lattices is **very** rapidly increasing in dimensions at least about 28.

Suppose L is unimodular, and dimension 25 (and if $\dim(L) < 25$, replace with $L \oplus \mathbb{Z}^{25-\dim(L)}$). Look at L^{even}, which is the sublattice of vectors of even norm in L. L^{even} has index 2 in L, so $(L^{\text{even}})'/L^{\text{even}}$ is cyclic of order 4. Look at a norm 4 vector v of $II_{1,25}$. $v^\perp$ has dimension 25, is negative definite, and $(v^\perp)'/v^\perp$ is cyclic of order 4. In fact, this gives bijections between:

1. orbits of norm 4 vectors in $II_{1,25}$
2. even lattices M of dimension 25 such that $M'/M \cong \mathbb{Z}/4$
3. 25 dimensional unimodular lattices

Orbits of negative norm vectors in $II_{1,25}$ are not interesting. Any two such vectors are conjugate if they have the same norm, and are equal multiples of primitive vectors.

For norm 0 vectors, we have a bijection between primitive norm 0 vectors and Niemeier lattices $w^\perp/w$. There are $24 \times \mathbb{Z}_{>0}$ orbits, given by nw, with w primitive and n a positive integer.

For positive norm vectors, we have 121 orbits of norm 2 vectors, 665 orbits of norm 4 vectors, and about 3000 orbits of norm 6 vectors.

Suppose v is a positive norm vector in $II_{1,25}$. We may assume v is in a fundamental domain of $II_{1,25}$. Then the simple roots of $v^\perp$ form a sublattice of Λ, taken as a subset of the simple roots of $II_{1,25}$. There are two possibilities:

1. $v^\perp$ has roots. Pick a root $r \in v^\perp$. Then $v + r$ has norm $v^2 + r^2 = v^2 - 2$, so we can "reduce" v to a vector of smaller norm. With some effort, this process can be **reversed**: knowing vectors of norm $2m$ allows us to find those of norm $2m+2$ with roots orthogonal to them. This works for $II_{1,1+8n}$ for all n.

2. $v^\perp$ has no roots. Recall $II_{1,25}$ has a Weyl vector ρ such that $(\rho, r) = 1$ for all simple roots r. We know $(v, r) > 0$ for **all** simple roots, as v is in the fundamental domain. Thus, $(v - \rho, r) \geq 0$ for all simple roots r. In particular, $v - \rho$ has positive norm $= v^2 - 2(v, \rho) < v^2$, so we can recover v as $(v - \rho) + \rho$ and induct.

Example: Find all 25 dimensional unimodular lattices L with no norm 2 vectors. L corresponds to a norm 4 vector v in the fundamental domain of $II_{1,25}$. Norm 2 vectors correspond to norm -2 vectors of $v^\perp$, so there are no (simple) roots of $II_{1,25}$ orthogonal to v. Using case 2, $a := v - \rho$ is in the fundamental domain. We know $0 \le a^2 < v^2 = 4$, so **either** $a^2 = 0$ or $a^2 = 2$.

If $a^2 = 0$, then $(a, \rho) = 2$, and a corresponds to some Niemeier lattice $a^\perp/a$. The Coxeter number of the Niemeier lattice is $(a, \rho) = 2$ (because $a = \sum m_i r_i$ norm 0, $(r_i, \rho) = 1$ implies $(a, \rho) = \sum m_i =$ Coxeter number), so the Niemeier lattice is of type A_1^{24}. Anyway, a is determined (up to conjugacy), so $v = a + \rho$ is determined. The corresponding lattice $v^\perp$ turns out to be $\mathbb{Z} \oplus \Lambda_{\text{odd}}$, where Λ_{odd} is the "odd Leech lattice", an odd 24 dimensional unimodular lattice of minimal norm 3. It was discovered about 20 years before Λ, and I think Witt discovered it but I'm not sure.

The second possibility is that $a^2 = 2$ and $(a, \rho) = 1$. Vectors with specified inner product with ρ are very easy to classify: any such vector is of the form $n\rho + \rho'$, where $\rho'^2 = 0$ and $(\rho', \rho) = 1$. If we write $II_{1,25} = \Lambda + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $\rho = (0, 0, 1)$ and ρ' is conjugate to $(0, 1, 0)$. This if $a^2 = 2$, a is conjugate to $\rho + \rho'$. There is only one possibility: $v^\perp \cong \Lambda \oplus \mathbb{Z}$.

So the only 25 dimensional unimodular lattices with no norm 2 vectors are $\Lambda \oplus \mathbb{Z}$ and $\Lambda_{\text{odd}} \oplus \mathbb{Z}$.

Remark: In all dimensions ≥ 23 **other** than 25, there are unimodular lattices with no roots.

If L is a 25 dimensional unimodular lattice, then the Dynkin diagram of norm 2 roots of L is a sublattice of Λ. For example, if $L = \mathbb{Z}^{25}$, the Dynkin diagram of norm 2 vectors is D_{25}, so $D_{25} \subset \Lambda$. We saw this explicitly in $(70, 0, 1, 2, \ldots, 24)$.

How do you prove that Λ has covering radius $\sqrt{2}$?

One way is to do a huge calculation, which was how Conway, Parker and Sloane proved it.

Another way is to recall that Λ has covering radius $\le \sqrt{2}$, if and only if $II_{1,25}$ has a Weyl vector ρ, if and only if the Dynkin diagram of any Niemeier lattice has rank 0 or 24 and all components have the same Coxeter number. This was first proved by Niemeier, and a "clean proof" was found by Venkov.

Recall that if L is an even unimodular lattice, then its theta function

$$\theta_L(\tau) = \sum_{\lambda \in L} e^{2\pi i (\lambda^2/2)\tau}$$

is a modular form of level one and weight $\frac{\dim(L)}{2}$. We modify the function (first done by Hecke) to:

$$\sum_{\lambda \in L} p(\lambda) e^{2\pi i (\lambda^2/2)\tau}$$

where p is a harmonic polynomial. This gives a level one, weight $\frac{\dim(L)}{2} + \deg(p)$ form. Now, we take L to be a Neimeier lattice, and $p(\lambda) = (\lambda, \alpha)^2 - \frac{\lambda^2\alpha^2}{24}$, where α is any fixed vector in L. The 24 in the denominator is the dimension of L, and makes the polynomial harmonic.

Key point: The space of level one cusp forms of weight 14 is 0, so

$$\sum_{\lambda^2=2n} (\lambda, \alpha)^2 = \frac{1}{24} \times 2n \times \alpha^2 \times |L(2n)|$$

where the last term is the number of norm $2n$ vectors in L. For $n = 1$, we have $\sum_{\lambda^2=2} (\lambda, \alpha)^2 = \frac{\alpha^2}{12} \times$ # norm 2 vectors of L. This easily implies the above properties of the root system of L.

Exercise: We need the fact that if R is a **connected** root system of norm 2 vectors, and $\alpha \in R$, then the number of vectors having inner product

$$\begin{array}{ccc} 2 & & 1 \\ 1 & & 2n-4 \\ 0 & \text{with } \alpha \text{ is} & * \\ -1 & & 2n-4 \\ -2 & & 1 \end{array}$$