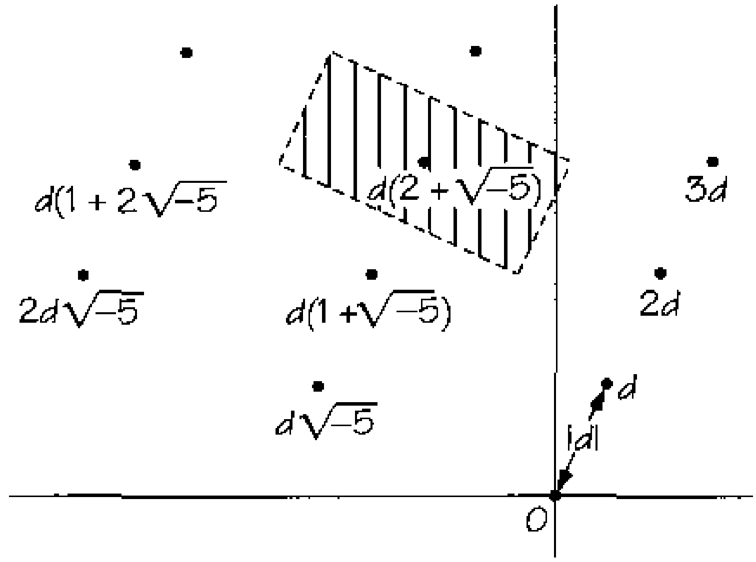


FIGURE 8.12 *Integer multiples of d in the number system $a + b\sqrt{-5}$*

THE NINE MAGIC DISCRIMINANTS

For exactly which negative numbers $-d$ does $\sqrt{-d}$ lead to a number system that has unique factorization into primes? The answer is now known; $-d$ must be one of the "Heegner numbers"

$$-1, -2, -3, -7, -11, -19, -43, -67, -163.$$

(In all except the first two cases we must allow, as integers, numbers $a + b\sqrt{-d}$ in which a and b are *halves* of integers, as we did for the Eisenstein integers.)

For a long time mathematicians were aware of these nine but were in the tantalizing position of knowing that there could be at most one more. The problem of deciding whether this outsider really existed was a notorious one, called the "tenth discriminant problem."

In 1936 Heilbronn and Linfoot showed that such a tenth d was bigger than 10^9. In 1952 Heegner published a proof that the list of nine was complete, but experts had some doubt about its validity. In 1966–67 two young mathematicians, Harold Stark in the United States and Alan Baker in Great Britain, independently obtained proofs, and the world was convinced. The story didn't really end here, because a year or two later Stark made a careful and detailed examination of

Heegner's proof and found that the critics had been unfair: the proof was essentially correct.

These numbers have many interesting properties. Euler discovered that the formula

$$n^2 - n + 41$$

gives the prime numbers

41 43 47 53 61 71 83 97 113 131 151 173 197 223 251
281 313 347 383 421 461 503 547 593 641 691 743 797 853
911 971 1033 1097 1163 1231 1301 1373 1447 1523 1601

when we set $n = 1, 2, 3, \ldots, 40$. What's the explanation?

The equation $x^2-x+41 = 0$ has solutions $x = \frac{1}{2}(1 \pm \sqrt{-163})$, and it can be shown that for a number $k > 1$, the formula

$$n^2 - n + k$$

represents primes for the consecutive numbers $n = 1, 2, \ldots, k - 1$ as long as $1-4k$ is one of the Heegner numbers. Now that we know them all, this leaves only the cases $k = 2, 3, 5, 11, 17$ and the one we've seen, 41.

Values for $n = 1, 2, \ldots, k - 1$

n^2-n+2	2
n^2-n+3	3 5
n^2-n+5	5 7 11 17
n^2-n+11	11 13 17 23 31 41 53 67 81 101
n^2-n+17	17 19 23 29 37 47 59 73 89 107 127 149 173 199 227 257

Another remarkable fact is that the numbers

$$e^{\pi\sqrt{43}} = 884736743.999777\ldots,$$

$$e^{\pi\sqrt{67}} = 147197952743.99999866\ldots,$$

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925007\ldots$$

are suspiciously close to integers. This is no mere accident! (The last one was part of Martin Gardner's famous April Fool hoax in 1975.) It can in fact be shown that for these numbers $X = e^{\pi\sqrt{d}}$, the formula

$$X - 744 + \frac{196884}{X} - \frac{21493760}{X^2} + \cdots$$

is exactly an integer and indeed a perfect cube! For the above values X is so large that the later terms are extremely small, so X itself must be nearly an integer.

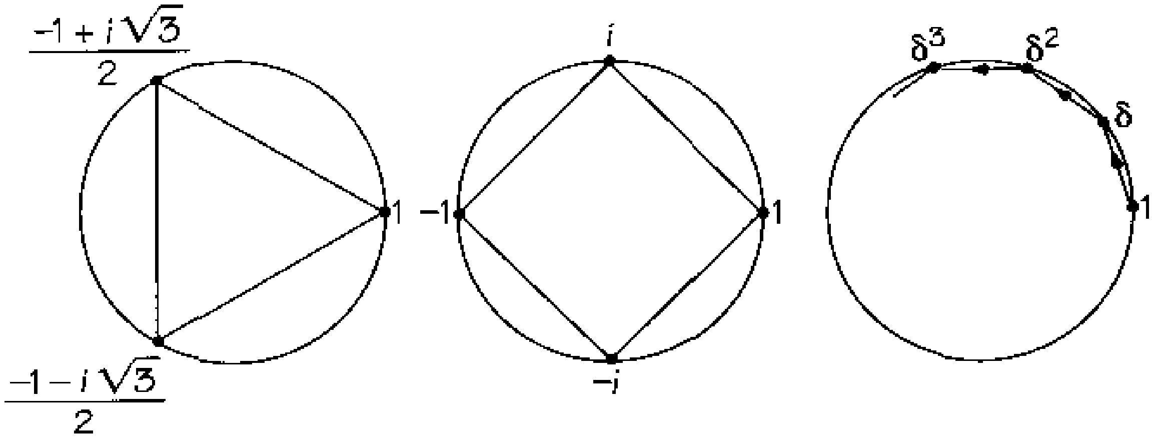
$$e^{\pi\sqrt{43}} = 960^3 + 744 - \text{(a bit)},$$
$$e^{\pi\sqrt{67}} = 5280^3 + 744 - \text{(a tiny bit)},$$
$$e^{\pi\sqrt{163}} = 640320^3 + 744 - \text{(a very tiny bit)!}$$

DE MOIVRE'S CIRCLE-CUTTING NUMBERS

Go to Mr. De Moivre; he knows these things better than I do.

Isaac Newton

Draw a regular polygon, centered at the origin in the complex plane, with one corner being the number 1 (Figure 8.13). What are the complex numbers corresponding to all the corners? These numbers were studied by the English mathematician Abraham De Moivre (1667–1754) long before it was realized that they had a geometrical meaning.



(a) (b) (c)

FIGURE 8.13 *De Moivre's cyclotomic numbers.*