6 Abel's Proof

Ideas, especially mathematical ideas, have a life that goes beyond their human creators. Without knowing Ruffini's work, Niels Henrik Abel (figure 6.1) discovered the same argument only a few years later and also gave the first essentially complete demonstration. Abel's story is one of the most moving in the history of mathematics. In his twenty-six years of life, he discovered whole new territories in mathematics, though struggling constantly with poverty and misunderstanding. His native Norway was then in its youth as an independent nation, and Abel's checkered fortunes to some extent mirrored Norway's struggle.

When Abel was 12, Norway separated from Denmark, long the dominant partner in their dual kingdom, and set up her own parliament, the Storting. But Norway could not completely separate from her more powerful neighbors. In 1814, Abel's father was among the delegates sent to offer the crown of Norway to the reigning Swedish king, Karl XIII. Abel's father was a pastor, like his father before him, and had a notable career as a member of the Storting. He was a man of the Enlightenment who read Voltaire and was active in the movement for literacy and vaccination in rural Norway. He wrote several popular catechisms and books of prayer

Figure 6.1
Niels Henrik Abel.

doubtless read by his son. He was interested in natural science and woke his children to see lunar eclipses. However, his political career ended ignominiously when he pressed unfounded accusations against certain powerful persons. He died an alcoholic, leaving nine children and a widow who also turned to alcohol for solace. After his funeral, she received visiting clergy while in bed with her peasant paramour.

Abel, then eighteen, found himself without support and obliged to act as the responsible adult for his younger siblings. Somehow, though, he continued his education. He had been fortunate to find a mentor in a young mathematics teacher, Berndt Michael Holmboe, who inspired him and became his lifelong friend. Abel soon showed an amazing ability to solve difficult problems. While still a high school student, he read Lagrange's work and Cauchy's 1815 paper on permutations. (Though Cauchy's paper was based on Ruffini's work, Abel did not read Ruffini, perhaps because his works were hard to find and in Italian.) Holmboe recognized and extolled his student's "excellent mathematical genius." In 1821, Abel entered the Royal Frederick's University, which had opened in 1819 in the new capital, Christiana (now called Oslo), then a city with only 11,000 inhabitants. He continued to make rapid strides, beginning to write original papers and going far beyond the skill of his teachers. With admirable understanding, those teachers proposed that he be granted a special fellowship to visit Paris and Berlin, though in fact he had little to learn even from the greatest mathematicians of the time.

While always modest, Abel during this time set himself to solve the most difficult and famous problems. In 1823, he tried his hand at Fermat's celebrated last theorem, to show the impossibility of finding integers a, b, and c that would satisfy the equation $a^n = b^n + c^n$, where n is an integer greater than 2. Not surprisingly, he found himself "at the end of my tether," as he put it, for this problem resisted solution until 1993, when it finally succumbed to very elaborate abstract techniques. Even so, he found what now are called "Abel's formulas," which showed that if any solutions existed, they would have to be extremely large numbers. Abel also turned his mind to the problem of the quintic equation. At first, he thought he had managed to solve it and was very excited. This was in 1821, twenty-two years after Ruffini had published the first version of his proof. At this point, Abel still did not know Ruffini's work, which is not surprising considering the isolation of Norway and its lack of mathematical libraries. Indeed, during the winter months, the Oslo fjord remained frozen and mail was often delayed.

Though Abel had probably taken note of Gauss's opinion that the quintic was unsolvable, he nonetheless refused to give up the search for a solution. After all, Gauss offered no proof for his assertion, and Abel, like Lagrange and most other mathematicians, could well consider that the search was not over. However, when his teachers asked him to give some numerical examples of his solution to the quintic, Abel soon realized that it was not as general as he had thought. This was a decisive moment, for he could have stubbornly resumed the quest for a fully general solution to repair the gaps in his work, on the assumption that a solution *had* to exist. Instead, he made an about-face and turned his efforts to proving the unsolvability of the quintic. He never explained his reasons, and one wonders what moved him. In 1824, two years after Ruffini's death, Abel published his first proof of the unsolvability of the quintic, which in many ways is close to Ruffini's proof, although it fills in an important gap that Ruffini had not noticed. In 1826, Abel read anonymous articles summarizing the work of Ruffini, which Abel acknowledged in his final (posthumous) paper: "The first, and if I am not mistaken the only one, who before me had tried to show the impossibility of the algebraic solution of general equations is the geometer *Ruffini*. But his paper is so complicated that it is very difficult to decide the correctness of his reasoning. It seems to me that his reasoning is not always satisfactory."

Because Ruffini's work overlaps largely with what Abel went on to do, I will not discuss it separately. I do not mean to belittle the recognition Ruffini is due. The theorem of unsolvability may better be called the Abel-Ruffini Theorem. Yet Abel's remark, as well as Lagrange's hesitation, indicate aspects of decisive proof in which Ruffini fell short. A proof that lacks a decisive step is not yet a proof. Accordingly, I will summarize Abel's version, indicating along the way the common elements in their arguments and the crucial gap in Ruffini's reasoning that Abel filled.

If you wish to read Abel's own words, appendix A contains a translation of his 1824 paper. Abel had this earliest version of the proof printed at his own expense as a pamphlet, hoping to use it as a "calling card" that would gain the attention of the great mathematicians, Gauss above all. However, to save paper and money, Abel compressed his argument to telegraphic terseness, which he amplified in his later accounts of the proof. Accordingly, I have added a running commentary to help the reader follow Abel's argument. Appendixes B and C fill out details he merely mentions. Here I will present the four crucial stages of the proof as Abel presented them in 1826, without spelling out all the technicalities to be found in the appendixes. In each case, I will present a summary statement, followed by my explanation.

Abel uses the time-honored method of *reductio ad absurdum*: he begins by assuming that the quintic is solvable and shows that this leads to a contradiction. His first step is to specify the form that a solution must have. Given a general equation of the mth degree (taking m as a prime number, so that it cannot be factored further),

$$a_m y^m + a_{m-1} y^{m-1} + a_{m-2} y^{m-2} + \cdots + a_2 y^2 + a_1 y + a_0 = 0, \quad (6.1)$$

Abel proves a very general statement about any solution, which he calls an "algebraic function":

(I) *All algebraic functions y can be expressed in the form*

$$y = p + R^{\frac{1}{m}} + p_2 R^{\frac{2}{m}} + \cdots + p_{m-1} R^{\frac{m-1}{m}}, \quad (6.2)$$

where $p, p_2, \ldots$ are finite sums of radicals and polynomials and $R^{\frac{1}{m}}$ is in general an irrational function of the coefficients of the original equation.

That is, if y is the solution of an algebraic equation of degree m, y can be expressed as a series of terms that contain nested roots involving the coefficients and irrational expressions like $R^{\frac{1}{m}}$. (Remember that $R^{\frac{1}{m}}$ is just another way of writing the mth root of R, $\sqrt[m]{R}$.) This is very close to the general form that Euler had already conjectured some years before. Abel gives a detailed proof (see appendix B, including some subtleties neglected here), but the idea can readily be illustrated with a few examples. As box 6.1 shows, the solution of the quadratic equation can be expressed as $y = p + R^{\frac{1}{2}}$, which is just the general form above with $m = 2$ and p a simple polynomial. Likewise, the solution of the cubic can be expressed as $y = p + R^{\frac{1}{3}} + p_2 R^{\frac{2}{3}}$, which follows the general form, with $m = 3$. In both of these cases, $m = 2$ and $m = 3$, m is a prime number. For the quartic, $m = 4$ is not prime and the general solution

Box 6.1
Abel's form for the quadratic equation

In the quadratic equation, $y^2 - a_1 y + a_0 = 0$, substitute $y = p + R^{\frac{1}{2}}$ to yield $(p^2 + R - a_1 p + a_0) + (2p - a_1)R^{\frac{1}{2}} = 0$. To satisfy this in general, each parenthesis must be separately zero (as Abel discusses in [A8] in appendix A), so $p = \frac{a_1}{2}$ and then $R = -a_0 + \frac{a_1^2}{4}$. Thus, $y = \frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}$, the familiar form of the quadratic solution. For the case of the cubic solution, see p. 174 in appendix B.

involves only combinations of the quadratic and cubic forms, as was discussed earlier.

In the case of the quintic, Abel's general form (6.2) becomes

$$y = p + R^{\frac{1}{5}} + p_2 R^{\frac{2}{5}} + p_3 R^{\frac{3}{5}} + p_4 R^{\frac{4}{5}}. \tag{6.3}$$

Because this follows from his general result about the form of the solution, either the solution of the quintic has this form, or there is no such solution. So Abel assumes hypothetically that the quintic does have a solution of exactly this form. He now goes on to show that this form leads to a contradiction. This requires three further steps.

The next step is crucial; Ruffini had assumed it without giving a proof, but Abel remedies this.

(II) *All algebraic functions y can be expressed in terms of rational functions of the roots of an equation.*

The idea here is simple but telling. In equation (6.3), the general form of y is expressed in terms of polynomials and also various irrational functions (here, $R^{\frac{1}{5}}, R^{\frac{2}{5}}, R^{\frac{3}{5}}, R^{\frac{4}{5}}$) of the *coefficients*. But Abel brilliantly proves that we can express

Box 6.2
The relation between roots and coefficients

Let $y = p + R^{\frac{1}{2}}$, as shown in box 6.1, which also shows that $R = -a_0 + \frac{a_1^2}{4}$. Now consider the two roots, $y_1 = \frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_0}$, $y_2 = \frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_0}$. Then $(y_1 - y_2) = \sqrt{a_1^2 - 4a_0}$ and thus $(y_1 - y_2)^2 = a_1^2 - 4a_0 = 4\left(-a_0 + \frac{a_1^2}{4}\right) = 4R$. The name *discriminant* is given to $a_1^2 - 4a_0$, which is zero when the roots are equal, $y_1 = y_2$, positive when the roots are real, and negative when they are imaginary.

y in terms of the *roots* instead of the original *coefficients* of the equation. What is important here is that all the various *irrational* functions of the coefficients that appear in y (e.g., $R^{\frac{1}{5}}$) are *rational* functions of the roots of the equation. To choose a simple example, in the quadratic equation above, whose general solution is $y = p + R^{\frac{1}{2}}$, box 6.2 shows that $4R = (y_1 - y_2)^2$, where y_1, y_2 are the two roots of the quadratic. So $2R^{\frac{1}{2}}$, the square root of $4R$, is equal to the difference of the two roots, $(y_1 - y_2)$. In agreement with Abel's proof, this is indeed a simple rational function of the roots.

In the case of the quintic, Abel's step II implies that $R^{\frac{1}{5}}$ is a rational function of the roots, as are its powers, $R^{\frac{2}{5}}$, $R^{\frac{3}{5}}$, $R^{\frac{4}{5}}$, as well as p, p_2, ... Because it is made up of products of all these rational functions, y is thus a rational function of the roots. Here there is an interesting tension: The roots are irrational functions of the *coefficients*, but those coefficients are always rational sums or products of the *roots*. This harks back to Girard's identities (see box 4.1), which showed that the coefficients were sums and products of the roots.

Let us return to the tension between the irrationality of the roots as functions of the coefficients, on the one hand, and the rationality of the coefficients as products of the roots, on the other. Roughly speaking, as the degree of the equation grows higher, it is harder and harder to reconcile this tension. Finally, with the quintic equation, it is no longer possible, and we cannot, in general, find a solution in radicals. However, this idea must be amplified much further.

The next step limits the hypothetical solution in a way that will prove to be decisive.

(III) *If a rational function of five quantities takes fewer than five values when the five quantities are permuted, it can take only two different values (equal in magnitude and opposite in sign), or one value, but never three or four values.*

Abel drew this theorem from the work of Cauchy, of which it is a special case. Cauchy's proof is presented in appendix C. It relies on looking at the different ways we can permute the roots of the equation. Abel also relies on the Fundamental Theorem of Algebra: A quintic equation must have at least one root and no more than five different roots. So there can be no more than five values for y, which, according to step II, is a rational function of the roots of the equation.

As before, let us consider any of the expressions in the solution that is a rational function of the roots, such as $R^{\frac{1}{5}}$. Cauchy's result requires that $R^{\frac{1}{5}}$ can take only one, two, or five values as the roots are permuted, but never three or four values. This gives Abel leverage to show that the solution cannot work. First, note that $R^{\frac{1}{5}}$ cannot, in general, take only one value, because then it could lead to a single solution, not five roots, which we have assumed to be unequal. Then, Abel investigates what happens if $R^{\frac{1}{5}}$ takes on five values (appendix A, [A15]–[A16]). This leads to a contradiction; he shows that $R^{\frac{1}{5}}$, having five possible values when the roots are permuted, would have to be equal to an expression having 120 values, which is impossible.

That excludes the possibility that there are five values for y. So Cauchy's result must require that there be only two (appendix A, [A27]). But this too leads to a contradiction, because when we switch the five roots around, Abel shows that we get an inconsistent result again. He derives an equation whose left-hand side has 120 possible values, while the right-hand side has only 10. Clearly, such an equation cannot be solved in general, and so the hypothetical solution leads to absurdities. Therefore Abel concludes that

(IV) *It is impossible to solve the general equation of the fifth degree in radicals.*

The strategy of Abel's argument is straightforward: he takes the only possible form a solution could have and shows that it leads to contradictory results when we permute the roots of the equation. This contradiction rests on a special property of the number five, shown by the number of values the hypothetical solution can take when subjected to permutations of its five roots. The argument also applies for degrees higher than five. For instance, we can multiply an unsolvable quintic by a factor of $y = y - 0$, and it would then be a sixth-degree equation that has one root $y = 0$ and five unsolvable roots, and similarly for any higher degree.

Though Abel's argument shows the impossibility of solving the quintic in general, it still seems opaque. The question remains: *why* this impossibility? To examine this further means we must seek the heart of Abel's proof.