CHAPTER IV.

The Construction of the Regular Polygon of 17 Sides.

1. We have just seen that the division of the circle into equal parts by the straight edge and compasses is possible only for the prime numbers studied by Gauss. It will now be of interest to learn how the construction can actually be effected.

The purpose of this chapter, then, will be to show in an elementary way how to inscribe in the circle the regular polygon of 17 sides.

Since we possess as yet no method of construction based upon considerations purely geometrical, we must follow the path indicated by our general discussions. We consider, first of all, the roots of the cyclotomic equation

$$x^{16} + x^{15} + \ldots + x^2 + x + 1 = 0,$$

and construct geometrically the expression, formed of square roots, deduced from it.

We know that the roots can be put into the transcendental form

$$\epsilon_\kappa = \cos \frac{2\kappa\pi}{17} + i \sin \frac{2\kappa\pi}{17} \quad (\kappa = 1, 2, \ldots 16)\ ;$$

and if

$$\epsilon_1 = \cos \frac{2\pi}{17} + i \sin \frac{2\pi}{17},$$

that

$$\epsilon_\kappa = \epsilon_1{}^\kappa.$$

Geometrically, these roots are represented in the complex plane by the vertices, different from 1, of the regular polygon of 17 sides inscribed in a circle of radius 1, having the origin as center. The selection of ϵ_1 is arbitrary, but for the construction it is essential to indicate some ϵ as the point of departure. Having fixed upon ϵ_1, the angle corresponding to ϵ_κ is κ times the angle corresponding to ϵ_1, which completely determines ϵ_κ.

2. The fundamental idea of the solution is the following: *Forming a primitive root to the modulus* 17 *we may arrange the* 16 *roots of the equation in a cycle in a determinate order.*

As already stated, a number a is said to be a primitive root to the modulus 17 when the congruence

$$a^s \equiv +1 \pmod{17}$$

has for least solution $s = 17 - 1 = 16$. The number 3 possesses this property; for we have

$$\left.\begin{array}{llll} 3^1 \equiv 3 & 3^5 \equiv 5 & 3^9 \equiv 14 & 3^{13} \equiv 12 \\ 3^2 \equiv 9 & 3^6 \equiv 15 & 3^{10} \equiv 8 & 3^{14} \equiv 2 \\ 3^3 \equiv 10 & 3^7 \equiv 11 & 3^{11} \equiv 7 & 3^{15} \equiv 6 \\ 3^4 \equiv 13 & 3^8 \equiv 16 & 3^{12} \equiv 4 & 3^{16} \equiv 1 \end{array}\right\} \pmod{17}.$$

Let us then arrange the roots ϵ_κ so that their indices are the preceding remainders in order

$$\epsilon_3,\ \epsilon_9,\ \epsilon_{10},\ \epsilon_{13},\ \epsilon_5,\ \epsilon_{15},\ \epsilon_{11},\ \epsilon_{16},\ \epsilon_{14},\ \epsilon_8,\ \epsilon_7,\ \epsilon_4,\ \epsilon_{12},\ \epsilon_2,\ \epsilon_6,\ \epsilon_1.$$

Notice that if r is the remainder of 3^κ (mod. 17), we have

$$3^\kappa = 17q + r,$$

whence
$$\epsilon_r = \epsilon_1^{\,r} = \epsilon_1^{\,3^\kappa}.$$

If r' is the next remainder, we have similarly

$$\epsilon_{r'} = \epsilon_1^{\,3^{\kappa+1}} = (\epsilon_1^{\,3^\kappa})^3 = (\epsilon_r)^3.$$

Hence in this series of roots each root is the cube of the preceding.

Gauss's method consists in decomposing this cycle into sums containing 8, 4, 2, 1 roots respectively, corresponding to the divisors of 16. Each of these sums is called a period.

The periods thus obtained may be calculated successively as roots of certain quadratic equations.

The process just outlined is only a particular case of that employed in the general case of the division into p equal parts. The $p-1$ roots of the cyclotomic equation are cyclically arranged by means of a primitive root of p, and the periods may be calculated as roots of certain auxiliary equations. The degree of these last depends upon the prime factors of $p-1$. They are not necessarily equations of the second degree.

The general case has, of course, been treated in detail by Gauss in his *Disquisitiones*, and also by Bachmann in his work, *Die Lehre von der Kreisteilung* (Leipzig, 1872).

3. In our case of the 16 roots the periods may be formed in the following manner: Form two periods of 8 roots by taking in the cycle, first, the roots of even order, then those of odd order. Designate these periods by x_1 and x_2, and replace each root by its index. We may then write symbolically

$$x_1 = 9 + 13 + 15 + 16 + 8 + 4 + 2 + 1,$$
$$x_2 = 3 + 10 + 5 + 11 + 14 + 7 + 12 + 6.$$

Operating upon x_1 and x_2 in the same way, we form 4 periods of 4 terms:

$$y_1 = 13 + 16 + 4 + 1,$$
$$y_2 = 9 + 15 + 8 + 2,$$
$$y_3 = 10 + 11 + 7 + 6,$$
$$y_4 = 3 + 5 + 14 + 12.$$

Operating in the same way upon the y's, we obtain 8 periods of 2 terms:

$$z_1 = 16 + 1, \qquad z_5 = 11 + 6,$$
$$z_2 = 13 + 4, \qquad z_6 = 10 + 7,$$
$$z_3 = 15 + 2, \qquad z_7 = 5 + 12,$$
$$z_4 = 9 + 8, \qquad z_8 = 3 + 14.$$

It now remains to show that *these periods can be calculated successively by the aid of square roots.*

4. It is readily seen that the sum of the remainders corresponding to the roots forming a period z is always equal to 17. These roots are then ϵ_r and ϵ_{17-r};

$$\epsilon_r = \cos r\frac{2\pi}{17} + i \sin r\frac{2\pi}{17},$$

$$\epsilon_{r'} = \epsilon_{17-r} = \cos (17-r)\frac{2\pi}{17} + i \sin (17-r)\frac{2\pi}{17},$$

$$= \cos r\frac{2\pi}{17} - i \sin r\frac{2\pi}{17}.$$

Hence

$$\epsilon_r + \epsilon_{r'} = 2 \cos r\frac{2\pi}{17}.$$

Therefore all the periods z are real, and we readily obtain

$$z_1 = 2 \cos \frac{2\pi}{17}, \qquad z_5 = 2 \cos 6\frac{2\pi}{17},$$

$$z_2 = 2 \cos 4\frac{2\pi}{17}, \qquad z_6 = 2 \cos 7\frac{2\pi}{17},$$

$$z_3 = 2 \cos 2\frac{2\pi}{17}, \qquad z_7 = 2 \cos 5\frac{2\pi}{17},$$

$$z_4 = 2 \cos 8\frac{2\pi}{17}, \qquad z_8 = 2 \cos 3\frac{2\pi}{17}.$$

Moreover, by definition,

$$x_1 = z_1 + z_2 + z_3 + z_4, \qquad x_2 = z_5 + z_6 + z_7 + z_8,$$

$$y_1 = z_1 + z_2, \quad y_2 = z_3 + z_4, \quad y_3 = z_5 + z_6, \quad y_4 = z_7 + z_8.$$

5. It will be necessary to determine the relative magnitude of the different periods. For this purpose we shall employ the following artifice: We divide the semicircle of unit radius into 17 equal parts and denote by $S_1, S_2, \ldots S_{17}$ the distances of the consecutive points of division $A_1, A_2, \ldots A_{17}$ from the initial point of the semicircle, S_{17} being equal to the diameter, *i.e.*, equal to 2. The angle $A_\kappa A_{17}O$ has the same measure as the half of the arc $A_\kappa O$, which equals $\frac{2\kappa\pi}{34}$. Hence

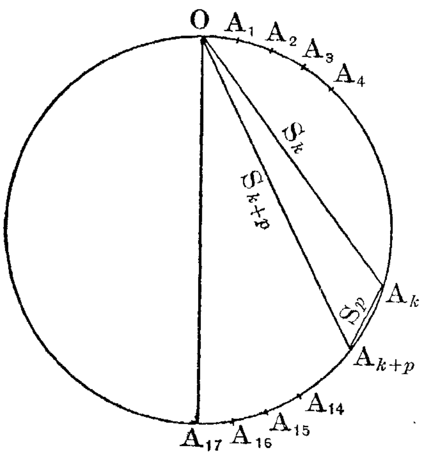

Fig. 3.

$$S_\kappa = 2\sin\frac{\kappa\pi}{34} = 2\cos\frac{(17-\kappa)\pi}{34}.$$

That this may be identical with $2\cos h\frac{2\pi}{17}$, we must have

$$4h = 17 - \kappa,$$
$$\kappa = 17 - 4h.$$

Giving to h the values 1, 2, 3, 4, 5, 6, 7, 8, we find for κ the values 13, 9, 5, 1, -3, -7, -11, -15. Hence

$$z_1 = S_{13}, \qquad z_5 = -S_7,$$
$$z_2 = S_1, \qquad z_6 = -S_{11},$$
$$z_3 = S_9, \qquad z_7 = -S_3,$$
$$z_4 = -S_{15}, \qquad z_8 = S_5.$$

The figure shows that S_κ increases with the index ; hence the order of increasing magnitude of the periods z is

$$z_4,\ z_6,\ z_5,\ z_7,\ z_2,\ z_8,\ z_3,\ z_1.$$

Moreover, the chord $A_\kappa A_{\kappa+p}$ subtends p divisions of the semicircumference and is equal to S_p ; the triangle $OA_\kappa A_{\kappa+p}$ shows that

$$S_{\kappa+p} < S_\kappa + S_p,$$

and *a fortiori*

$$S_{\kappa+p} < S_{\kappa+r} + S_{p+r}.$$

Calculating the differences two and two of the periods y, we easily find

$$
\begin{aligned}
y_1 - y_2 &= S_{13} + S_1 - S_9 + S_{15} > 0,\\
y_1 - y_3 &= S_{13} + S_1 + S_7 + S_{11} > 0,\\
y_1 - y_4 &= S_{13} + S_1 + S_3 - S_5 > 0,\\
y_2 - y_3 &= S_9 - S_{15} + S_7 + S_{11} > 0,\\
y_2 - y_4 &= S_9 - S_{15} + S_3 - S_5 < 0,\\
y_3 - y_4 &= - S_7 - S_{11} + S_3 - S_5 < 0.
\end{aligned}
$$

Hence

$$y_3 < y_2 < y_4 < y_1.$$

Finally we obtain in a similar way

$$x_2 < x_1.$$

6. We now propose to calculate $z_1 = 2 \cos \frac{2\pi}{17}$. After making this calculation and constructing z_1, we can easily deduce the side of the regular polygon of 17 sides. In order to find the quadratic equation satisfied by the periods, we proceed to determine symmetric functions of the periods.

Associating z_1 with the period z_2 and thus forming the period y_1, we have, first,

$$z_1 + z_2 = y_1.$$

Let us now determine $z_1 z_2$. We have

$$z_1 z_2 = (16 + 1)\ (13 + 4),$$

where the symbolic product κp represents

$$\epsilon_\kappa \cdot \epsilon_p = \epsilon_{\kappa + p}.$$

Hence it should be represented symbolically by $\kappa + p$, remembering to subtract 17 from $\kappa + p$ as often as possible. Thus,

$$z_1 z_2 = 12 + 3 + 14 + 5 = y_4.$$

Therefore z_1 and z_2 are the roots of the quadratic equation

$$(\zeta) \qquad z^2 - y_1 z + y_4 = 0,$$

whence, since $z_1 > z_2$,

$$z_1 = \frac{y_1 + \sqrt{y_1^2 - 4y_4}}{2}, \qquad z_2 = \frac{y_1 - \sqrt{y_1^2 - 4y_4}}{2}.$$

We must now determine y_1 and y_4. Associating y_1 with the period y_2, thus forming the period x_1, and y_3 with the period y_4, thus forming the period x_2, we have, first,

$$y_1 + y_2 = x_1.$$

Then,

$$y_1 y_2 = (13 + 16 + 4 + 1)(9 + 15 + 8 + 2).$$

Expanding symbolically, the second member becomes equal to the sum of all the roots; that is, to -1. Therefore y_1 and y_2 are the roots of the equation

$$(\eta) \qquad y^2 - x_1 y - 1 = 0,$$

whence, since $y_1 > y_2$,

$$y_1 = \frac{x_1 + \sqrt{x_1^2 + 4}}{2}, \qquad y_2 = \frac{x_1 - \sqrt{x_1^2 + 4}}{2}.$$

Similarly,

$$y_3 + y_4 = x_2$$

and

$$y_3 y_4 = -1.$$

Hence y_3 and y_4 are the roots of the equation

$$(\eta') \qquad y^2 - x_2 y - 1 = 0;$$

whence, since $y_4 > y_3$,

$$y_4 = \frac{x_2 + \sqrt{x_2^2 + 4}}{2}, \qquad y_3 = \frac{x_2 - \sqrt{x_2^2 + 4}}{2}.$$

It now remains to determine x_1 and x_2. Since $x_1 + x_2$ is equal to the sum of all the roots,

$$x_1 + x_2 = -1.$$

Further,

$$x_1 x_2 = (13 + 16 + 4 + 1 + 9 + 15 + 8 + 2)$$
$$(10 + 11 + 7 + 6 + 3 + 5 + 14 + 12).$$

Expanding symbolically, each root occurs 4 times, and thus

$$x_1 x_2 = -4.$$

Therefore x_1 and x_2 are the roots of the quadratic

$$(\xi) \qquad x^2 + x - 4 = 0;$$

whence, since $x_1 > x_2$,

$$x_1 = \frac{-1+\sqrt{17}}{2}, \qquad x_2 = \frac{-1-\sqrt{17}}{2}.$$

Solving equations ξ, η, η', ζ in succession, z_1 is determined by a series of square roots.

Effecting the calculations, we see that z_1 depends upon the four square roots

$$\sqrt{17},\ \sqrt{x_1^2+4},\ \sqrt{x_2^2+4},\ \sqrt{y_1^2-4y_4}.$$

If we wish to reduce z_1 to the normal form we must see whether any one of these square roots can be expressed rationally in terms of the others.

Now, from the roots of (η),

$$\sqrt{x_1^2+4} = y_1 - y_2,$$
$$\sqrt{x_2^2+4} = y_4 - y_3.$$

Expanding symbolically, we verify that

$$(y_1 - y_2)(y_4 - y_3) = 2(x_1 - x_2),*$$

* $(y_1 - y_2)(y_4 - y_3) = (13 + 16 + 4 + 1 - 9 - 15 - 8 - 2)(3 + 5 + 14 + 12 - 10 - 11 - 7 - 6)$

$$\begin{aligned}
&= 16 + 1 + 10 + 8 - 6 - 7 - 3 - 2\\
&+ 2 + 4 + 13 + 11 - 9 - 10 - 6 - 5\\
&+ 7 + 9 + 1 + 16 - 14 - 15 - 11 - 10\\
&+ 4 + 6 + 15 + 13 - 11 - 12 - 8 - 7\\
&- 12 - 14 - 6 - 4 + 2 + 3 + 16 + 15\\
&- 1 - 3 - 12 - 10 + 8 + 9 + 5 + 4\\
&- 11 - 13 - 5 - 3 + 1 + 2 + 15 + 14\\
&- 5 - 7 - 16 - 14 + 12 + 13 + 9 + 8
\end{aligned}$$

$= 2(16 + 1 + 8 + 2 + 4 + 13 + 15 + 9 - 10 - 6 - 7 - 3 - 11 - 5 - 14 - 12)$

$= 2(x_1 - x_2).$

that is,

$$\sqrt{x_1^2+4}\ \sqrt{x_2^2+4}=2\sqrt{17}.$$

Hence $\sqrt{x_2^2+4}$ can be expressed rationally in terms of the other two square roots. This equation shows that if two of the three differences y_1-y_2, y_4-y_3, x_1-x_2 are positive, the same is true of the third, which agrees with the results obtained directly.

Replacing now x_1, y_1, y_4 by their numerical values, we obtain in succession

$$x_1=\frac{-1+\sqrt{17}}{2},$$

$$y_1=\frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{4},$$

$$y_4=\frac{-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}}{4},$$

$$z_1=\frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{8}$$

$$+\frac{\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}}{8}.$$

The algebraic part of the solution of our problem is now completed. We have already remarked that there is no known construction of the regular polygon of 17 sides based upon purely geometric considerations. There remains, then, only the geometric translation of the individual algebraic steps.

7. We may be allowed to introduce here a brief historical account of geometric constructions with straight edge and compasses.

In the geometry of the ancients the straight edge and compasses were always used together; the difficulty lay merely in bringing together the different parts of the figure so as not to draw any unnecessary lines. Whether the several steps in the construction were made with straight edge or with compasses was a matter of indifference.

On the contrary, in 1797, the Italian Mascheroni succeeded in effecting all these constructions with the compasses alone; he set forth his methods in his *Geometria del compasso*, and claimed that constructions with compasses were practically more exact than those with the straight edge. As he expressly stated, he wrote for mechanics, and therefore with a practical end in view. Mascheroni's original work is difficult to read, and we are under obligations to Hutt for furnishing a brief *résumé* in German, *Die Mascheroni'schen Constructionen* (Halle, 1880).

Soon after, the French, especially the disciples of Carnot, the author of the *Géométrie de position*, strove, on the other hand, to effect their constructions as far as possible with the straight edge. (See also Lambert, *Freie Perspective*, 1774.)

Here we may ask a question which algebra enables us to answer immediately: In what cases can the solution of an algebraic problem be constructed with the straight edge alone? The answer is not given with sufficient explicitness by the authors mentioned. We shall say:

With the straight edge alone we can construct all algebraic expressions whose form is rational.

With a similar view Brianchon published in 1818 a paper, *Les applications de la théorie des transversales*, in which he shows how his constructions can be effected in many cases with the straight edge alone. He likewise insists upon the practical value of his methods, which are especially adapted to field work in surveying.

Poncelet was the first, in his *Traité des propriétés projectives* (Vol. I, Nos. 351–357), to conceive the idea that it is sufficient to use a *single fixed circle* in connection with the straight lines of the plane in order to construct all expressions depending upon square roots, the center of the fixed circle being given.

This thought was developed by Steiner in 1833 in a celebrated paper entitled *Die geometrischen Constructionen, ausgeführt mittels der geraden Linie und eines festen Kreises, als Lehrgegenstand für höhere Unterrichtsanstalten und zum Selbstunterricht.*

8. To construct the regular polygon of 17 sides we shall follow the method indicated by von Staudt (Crelle's *Journal*, Vol. XXIV, 1842), modified later by Schröter (Crelle's *Journal*, Vol. LXXV, 1872). The construction of the regular polygon of 17 sides is made in accordance with the methods indicated by Poncelet and Steiner, inasmuch as besides the straight edge but one fixed circle is used.*

First, we will show *how with the straight edge and one fixed circle we can solve every quadratic equation.*

At the extremities of a diameter of the fixed unit circle (Fig. 4) we draw two tangents, and select the lower as the axis of X, and the diameter perpendicular to it as the axis of Y. Then the equation of the circle is

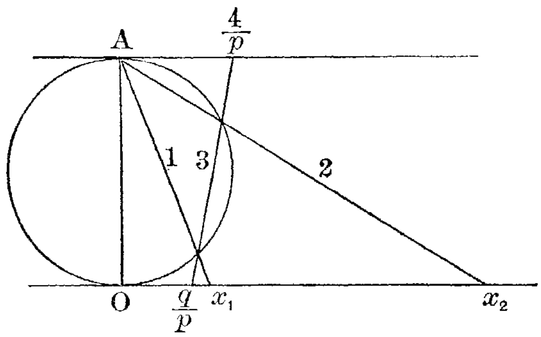

FIG. 4.

$$x^2 + y(y-2) = 0.$$

Let

$$x^2 - px + q = 0$$

be any quadratic equation with real roots x_1 and x_2. Required to construct the roots x_1 and x_2 upon the axis of X.

Lay off upon the upper tangent from A to the right, a segment measured by $\frac{4}{p}$; upon the axis of X from O, a segment

* A Mascheroni construction of the regular polygon of 17 sides by L. Gérard is given in *Math. Annalen*, Vol. XLVIII, 1896, pp. 390–392.

measured by $\frac{q}{p}$; connect the extremities of these segments by the line 3 and project the intersections of this line with the circle from A, by the lines 1 and 2, upon the axis of X. The segments thus cut off upon the axis of X are measured by x_1 and x_2.

Proof. Calling the intercepts upon the axis of X, x_1 and x_2, we have the equation of the line 1,

$$2x + x_1(y-2) = 0\,;$$

of the line 2,

$$2x + x_2(y-2) = 0.$$

If we multiply the first members of these two equations we get

$$x^2 + \frac{x_1+x_2}{2}\,x\,(y-2) + \frac{x_1x_2}{4}\,(y-2)^2 = 0$$

as the equation of the line pair formed by 1 and 2. Subtracting from this the equation of the circle, we obtain

$$\frac{x_1+x_2}{2}\,x\,(y-2) + \frac{x_1x_2}{4}\,(y-2)^2 - y\,(y-2) = 0.$$

This is the equation of a conic passing through the four intersections of the lines 1 and 2 with the circle. From this equation we can remove the factor $y-2$, corresponding to the tangent, and we have left

$$\frac{x_1+x_2}{2}\,x + \frac{x_1x_2}{4}\,(y-2) - y = 0,$$

which is the equation of the line 3. If we now make $x_1 + x_2 = p$ and $x_1x_2 = q$, we get

$$\frac{p}{2}\,x + \frac{q}{4}\,(y-2) - y = 0,$$

and the transversal 3 cuts off from the line $y = 2$ the segment $\frac{4}{p}$, and from the line $y=0$ the segment $\frac{q}{p}$. Thus the correctness of the construction is established.

9. In accordance with the method just explained, we shall now construct the roots of our four quadratic equations. They are (see pp. 29–31)

(ξ) $x^2 + x - 4 = 0$, with roots x_1 and x_2; $x_1 > x_2$,
(η) $y^2 - x_1 y - 1 = 0$, with roots y_1 and y_2; $y_1 > y_2$,
(η') $y^2 - x_2 y - 1 = 0$, with roots y_3 and y_4; $y_4 > y_3$,
(ζ) $z^2 - y_1 z + y_4 = 0$, with roots z_1 and z_2; $z_1 > z_2$.

These will furnish

$$z_1 = 2 \cos \frac{2\pi}{17},$$

whence it is easy to construct the polygon desired. We notice further that to construct z_1 it is sufficient to construct x_1, x_2, y_1, y_4.

We then lay off the following segments: upon the upper tangent, $y=2$,

$$-4,\ \frac{4}{x_1},\ \frac{4}{x_2},\ \frac{4}{y_1};$$

upon the axis of X,

$$+4,\ -\frac{1}{x_1},\ -\frac{1}{x_2},\ \frac{y_4}{y_1}.$$

This may all be done in the following manner: The straight line connecting the point $+4$ upon the axis of X with the point -4 upon the tangent $y=2$ cuts the circle in

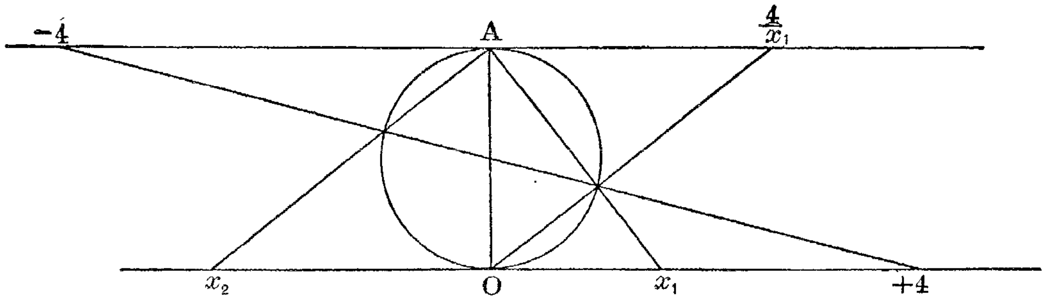

FIG. 5.

two points, the projection of which from the point A (0, 2), the upper vertex of the circle, gives the two roots x_1, x_2 of the first quadratic equation as intercepts upon the axis of X.

To solve the second equation we have to lay off $\frac{4}{x_1}$ above and $-\frac{1}{x_1}$ below.

To determine the first point we connect x_1 upon the axis of X with A, the upper vertex, and from O, the lower vertex, draw another straight line through the intersection of this line with the circle. This cuts off upon the upper tangent the intercept $\frac{4}{x_1}$. This can easily be shown analytically.

The equation of the line from A to x_1 (Fig. 5),

$$2x + x_1 y = 2x_1,$$

and that of the circle,

$$x^2 + y\,(y - 2) = 0,$$

give as the coördinates of their intersection

$$\frac{4x_1}{x_1^2 + 4},\ \frac{2x_1^2}{x_1^2 + 4}.$$

The equation of the line from O through this point becomes

$$y = \frac{x_1}{2}\,x,$$

cutting off upon $y = 2$ the intercept $\frac{4}{x_1}$.

We reach the same conclusion still more simply by the use of some elementary notions of projective geometry. By our construction we have obviously associated with every point x of the lower range one, and only one, point of the upper, so that to the point $x = \infty$ corresponds the point $x' = 0$, and conversely. Since in such a correspondence there must exist a linear relation, the abscissa x' of the upper point must satisfy the equation

$$x' = \frac{\text{const.}}{x}.$$

Since $x' = 2$ when $x = 2$, as is obvious from the figure, the constant $= 4$.

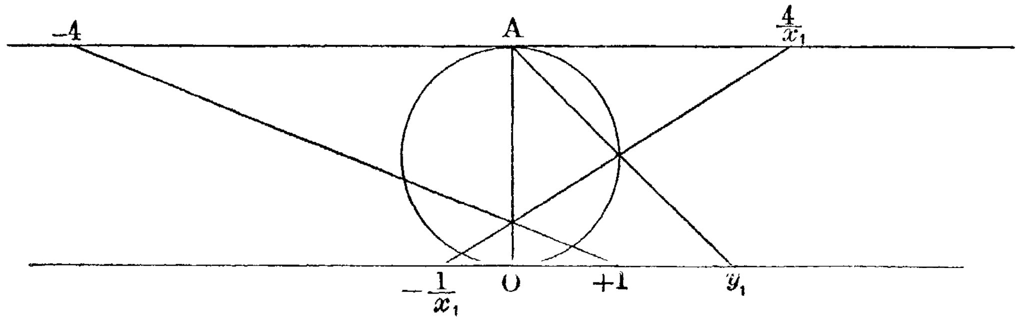

FIG. 6.

To determine $-\frac{1}{x_1}$ upon the axis of X we connect the point -4 upon the upper with the point $+1$ upon the lower tangent (Fig. 6). The point thus determined upon the vertical diameter we connect with the point $\frac{4}{x_1}$ above. This line cuts off upon the axis of X the intercept $-\frac{1}{x_1}$. For the line from -4 to $+1$,

$$5y + 2x = 2,$$

intersects the vertical diameter in the point $(0, \frac{2}{5})$. Hence the equation of the line from $\frac{4}{x_1}$ to this point is

$$5y - 2x_1x = 2,$$

and its intersection with the lower tangent gives $-\frac{1}{x_1}$.

The projection from A of the intersections of the line from $-\frac{1}{x_1}$ to $\frac{4}{x_1}$ with the circle determines upon the axis of X the two roots of the second quadratic equation, of which, as already noted, we need only the greater, y_1. This corresponds, as shown by the figure, to the projection of the upper intersection of our transversal with the circle.

Similarly, we obtain the roots of the third quadratic equation. Upon the upper tangent we project from O the intersection of the circle with the straight line which gave upon the axis of X the root $+x_2$. This immediately gives the intercept $\frac{4}{x_2}$, by reason of the correspondence just explained.

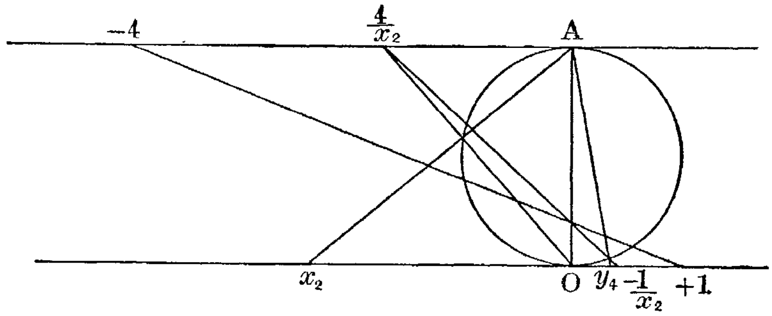

FIG. 7.

If we connect this point with the point where the vertical diameter intersects the line joining -4 above and $+1$ below, we cut off upon the axis of X the segment $-\frac{1}{x_2}$, as desired. If we project that intersection of this transversal with the circle which lies in the positive quadrant from A upon the axis of X, we have constructed the required root y_4 of the third quadratic equation.

We have finally to determine the root z_1 of the fourth quadratic equation and for this purpose to lay off $\frac{4}{y_1}$ above and $\frac{y_4}{y_1}$ below. We solve the first problem in the usual way, by projecting the intersection of the circle with the line connecting A with $+y_1$ below, from O upon the upper tangent, thus obtaining $\frac{4}{y_1}$. For the other segment we connect the point $+4$ above with y_4 below, and then the point thus determined upon the vertical diameter produced with $\frac{4}{y_1}$. This line cuts off upon the axis of X exactly the segment desired, $\frac{y_4}{y_1}$. For the line a (Fig. 8) has the equation

$$(y_4 - 4)\,y + 2x = 2y_4.$$

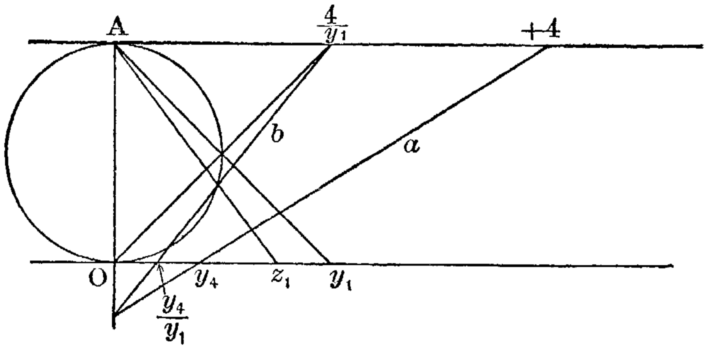

FIG. 8.

It cuts off upon the vertical diameter the segment $\frac{2y_4}{y_4 - 4}$. The equation of the line b is then

$$2y_1x + (y_4 - 4)\,y = 2y_4,$$

and its intersection with the axis of X has the abscissa $\frac{y_4}{y_1}$.

If we project the upper intersection of the line b with the circle from A upon the axis of X, we obtain $z_1 = 2\cos\frac{2\pi}{17}$. If we desire the simple cosine itself we have only to draw a diameter parallel to the axis of X, on which our last projecting ray cuts off directly $\cos\frac{2\pi}{17}$. A perpendicular erected at this point gives immediately the first and sixteenth vertices of the regular polygon of 17 sides.

The period z_1 was chosen arbitrarily; we might construct in the same way every other period of two terms and so find the remaining cosines. These constructions, made on separate figures so as to be followed more easily, have been combined in a single figure (Fig. 9), which gives the complete construction of the regular polygon of 17 sides.

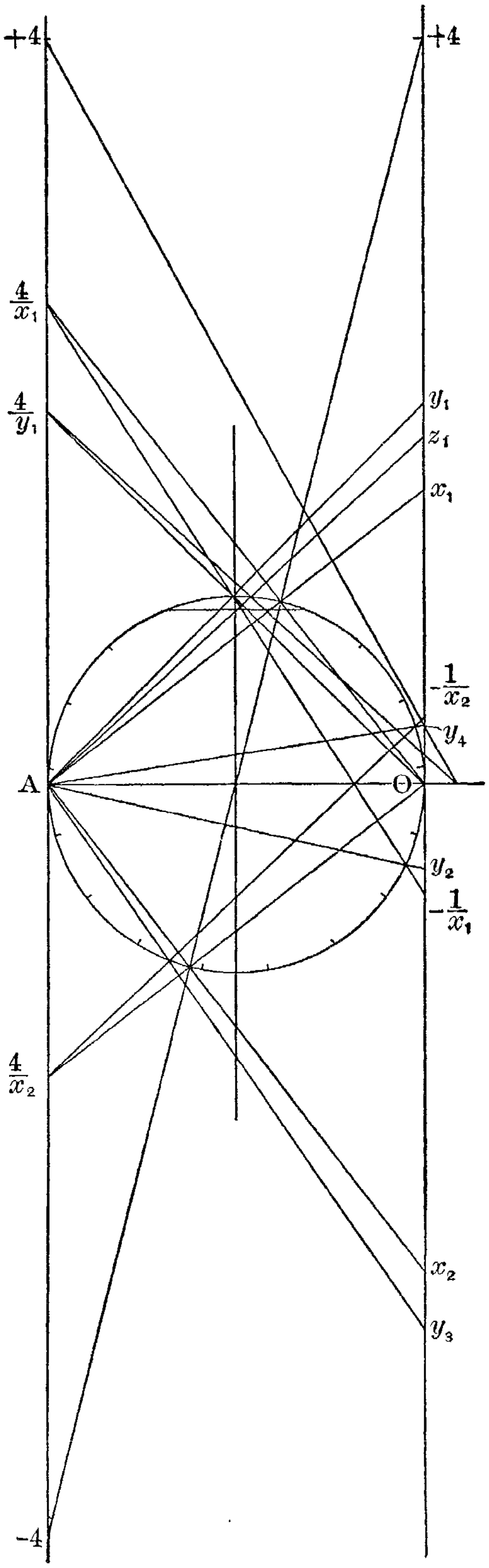
+4
+4
$\frac{4}{x_1}$
$\frac{4}{y_1}$
y_1
z_1
x_1
$-\frac{1}{x_2}$
y_4
A
O
y_2
$-\frac{1}{x_1}$
$\frac{4}{x_2}$
x_2
y_3
-4

FIG. 9.