APPENDIX A

DETERMINATION OF ALL FINITE GROUPS OF PROPER ROTATIONS IN 3-SPACE (cf. p. 77).

A SIMPLE PROOF for the completeness of the list (5) in Lecture II is based on the fact first established by Leonhard Euler in the eighteenth century that every proper rotation in 3-space which is not the identity I is rotation around an axis, i.e. it leaves fixed not only the origin O but every point on a certain straight line through O, the axis l. It is sufficient to consider the two-dimensional sphere Σ of unit radius around O instead of the three-dimensional space; for every rotation carries Σ into itself and thus is a one-to-one mapping of Σ into itself. Every proper rotation $\neq I$ has two fixed points on Σ which are antipodes of each other, namely the points where the axis l pierces the sphere.

Given a finite group Γ of proper rotations of order N, we consider the fixed points of the $N - 1$ operations of Γ which are different from I. We call them poles. Each pole p has a definite multiplicity ν ($= 2$ or 3 or 4 or $\cdots$): The operations S of our group which leave p invariant consist of the iterations of the rotation around the corresponding axis by $360°/\nu$, and hence there are exactly ν such operations S. They form a cyclic subgroup Γ_p of order ν. One of these operations is the identity, hence the number of operations $\neq I$ leaving p fixed amounts to $\nu - 1$.

For any point p on the sphere we may consider the finite set C of those points q into which p is carried by the operations of the group; we call them points equivalent to p.

Because Γ is a group this equivalence is of the nature of an equality, i.e. the point p is equivalent to itself; if q is equivalent to p then p is equivalent to q; and if both q_1 and q_2 are equivalent to p then q_1 and q_2 are equivalent among each other. We speak of our set as a *class* of equivalent points; any point of the class may serve as its representative p inasmuch as the class contains with p all the points equivalent to p and no others. While the points of a sphere are indiscernible under the group of all proper rotations, the points of a class remain even indiscernible after this group has been limited to the finite subgroup Γ.

Of how many points does the class C_p of the points equivalent to p consist? The answer: of N points, that naturally suggests itself, is correct provided I is the only operation of the group which leaves p fixed. For then any two different operations S_1,S_2 of Γ carry p into two different points $q_1 = pS_1$, $q_2 = pS_2$ since their coincidence $q_1 = q_2$ would imply that the operation $S_1S_2^{-1}$ carries p into itself, and would thus lead to $S_1S_2^{-1} = I$, $S_1 = S_2$. But suppose now that p is a pole of multiplicity ν so that ν operations of the group carry p into itself. Then, I maintain, the number of points q of which the class C_p consists equals N/ν.

Proof: Since the points of the class are indiscernible even under the given group Γ, each must be of the same multiplicity ν. Let us first demonstrate this explicitly. If the operation L of Γ carries p into q then $L^{-1}SL$ carries q into q provided S carries p into p. Vice versa, if T is any operation of Γ carrying q into itself then $S = LTL^{-1}$ carries p into p and hence T is of the form $L^{-1}SL$ where S is an element of the group Γ_p.

Thus if $S_1 = I$, $S_2, \cdots, S_\nu$ are the ν elements leaving p fixed then

$$T_1 = L^{-1}S_1L, \qquad T_2 = L^{-1}S_2L, \cdots,$$
$$T_\nu = L^{-1}S_\nu L$$

are the ν different operations leaving q fixed. Moreover, the ν different operations $S_1L, \cdots, S_\nu L$ carry p into q. Vice versa, if U is an operation of Γ carrying p into q then UL^{-1} carries p into p and thus is one of the operations S leaving p fixed; therefore $U = SL$ where S is one of the ν operations $S_1, \cdots, S_\nu$. Now let $q_1, \cdots, q_n$ be the n different points of the class $C = C_p$ and let L_i be one of the operations in Γ carrying p into q_i ($i = 1, \cdots, n$). Then all the $n \cdot \nu$ operations of the table

$$\begin{array}{l} S_1L_1, \cdots, S_\nu L_1, \\ S_1L_2, \cdots, S_\nu L_2, \\ \cdots\cdots\cdots\cdots \\ S_1L_n, \cdots, S_\nu L_n \end{array}$$

are different from each other. Indeed each individual line consists of different operations. And all the operations of, say, the second line must be different from those in the fifth line since the former carry p into q_2 and the latter into the point $q_5 \neq q_2$. Moreover every operation of the group Γ is contained in the table because any one of them carries p into one of the points $q_1, \cdots, q_n$, say into q_i, and must therefore figure in the ith line of our table.

This proves the relation $N = n\nu$ and thus the fact that the multiplicity ν is a divisor of N. We use the notation $\nu = \nu_p$ for the multiplicity of a pole p; we know that it is the same for every pole p in a given class C, and it can therefore also be denoted in an unambiguous manner by ν_C. The multiplicity ν_C and the number n_C of poles in the class C are connected by the relation $n_C \nu_C = N$.

After these preparations let us now consider all pairs (S, p) consisting of an operation $S \neq I$ of the group Γ and a point p left fixed by S—or, what is the same, of any pole p and any operation $S \neq I$ of the group leaving p fixed. This double description indicates a double enumeration of those pairs. On the one hand there are $N - 1$ operations S in the group that are different from I, and each has two antipodic fixed points; hence the number of the pairs equals $2(N - 1)$. On the other hand, for each pole p there are $\nu_p - 1$ operations $\neq I$ in the group leaving p fixed, and hence the number of the pairs equals the sum

$$\sum_p (\nu_p - 1)$$

extending over all poles p. We collect the poles into classes C of equivalent poles and thus obtain the following basic equation:

$$2(N - 1) = \sum_C n_C(\nu_C - 1)$$

where the sum to the right extends over all classes C of poles. On taking the equation $n_C \nu_C = N$ into account, division by N yields the relation

$$2 - \frac{2}{N} = \sum_C \left(1 - \frac{1}{\nu_C}\right).$$

What follows is a discussion of this equation.

The most trivial case is the one in which the group Γ consists of the identity only; then $N = 1$, and there are no poles.

Leaving aside this trivial case we can say that N is at least 2 and hence the left side of our equation is at least 1, but less than 2. The first fact makes it impossible for the sum to the right to consist of one term only. Hence there are at least two classes C. But certainly not more than 3. For as each ν_C is at least 2, the sum to the right would at least be 2 if it consisted of 4 or more terms. Consequently we have either two or three classes of equivalent poles (Cases II and III respectively).

II. In this case our equation gives

$$\frac{2}{N} = \frac{1}{\nu_1} + \frac{1}{\nu_2} \qquad \text{or} \qquad 2 = \frac{N}{\nu_1} + \frac{N}{\nu_2}.$$

But two positive integers $n_1 = N/\nu_1$, $n_2 = N/\nu_2$ can have the sum 2 only if each equals 1:

$$\nu_1 = \nu_2 = N; \qquad n_1 = n_2 = 1.$$

Hence each of the two classes of equivalent poles consists of *one* pole of multiplicity N. What we find here is the cyclic group of rotations around a (vertical) axis of order N.

III. In this case we have

$$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} = 1 + \frac{2}{N}.$$

Arrange the multiplicities ν in ascending order, $\nu_1 \leq \nu_2 \leq \nu_3$. Not all three numbers ν_1, ν_2, ν_3 can be greater than 2; for then the left side would give a result that is $\leq 1/3 + 1/3 + 1/3 = 1$, contrary to the value of the right side. Hence $\nu_1 = 2$,

$$\frac{1}{\nu_2} + \frac{1}{\nu_3} = \frac{1}{2} + \frac{2}{N}.$$

Not both numbers ν_2, ν_3 can be ≥ 4, for then the left sum would be $\leq 1/2$. Therefore $\nu_2 = 2$ or 3.

First alternative III_1: $\nu_1 = \nu_2 = 2$,
$$N = 2\nu_3.$$

Second alternative III_2: $\nu_1 = 2$, $\nu_2 = 3$;
$$\frac{1}{\nu_3} = \frac{1}{6} + \frac{2}{N}.$$

Set $\nu_3 = n$ in Case III_1. We have two classes of poles of multiplicity 2 each consisting of n poles, and one class consisting of two poles of multiplicity n. It is easily seen that these conditions are fulfilled by the dihedral group D'_n and by this group only.

For the second alternative III_2 we have, in view of $\nu_3 \geq \nu_2 = 3$, the following three possibilities:

$$\nu_3 = 3, \quad N = 12; \qquad \nu_3 = 4, \quad N = 24;$$
$$\nu_3 = 5, \quad N = 60,$$

which we denote by T, W, P respectively.

T: There are two classes of 4 three-poles each. It is clear that the poles of one class must form a regular tetrahedron and those of the other are their antipodes. We therefore obtain the tetrahedral group. The 6 equivalent two-poles are the projections from O onto the sphere of the centers of the 6 edges.

W: One class of 6 four-poles, forming the corners of a regular octahedron; hence the octahedral group. One class of 8 three-poles (corresponding to the centers of the sides); one class of 12 two-poles (corresponding to the centers of the edges).

Case P: One class of 12 five-poles which must form the corners of a regular icosahedron. The 20 three-poles correspond to the centers of the 20 sides, the 30 two-poles to the centers of the 30 edges of the polyhedron.