$$(f[\alpha]_k)(\tau) = \sum_{n=0}^{\infty} a_n q_{h'}^n, \quad q_{h'} = e^{2\pi i\tau/h'}$$

then since $e^{2\pi i(\tau+j)/h'} = \mu_{h'}^j q_{h'}$ where $\mu_{h'} = e^{2\pi i/h'}$ is the complex $h'$th root of unity, it follows that

$$(f[\pm\alpha\beta]_k)(\tau) = (\pm 1)^k \sum_{n=0}^{\infty} a_n \mu_{h'}^{nj} q_{h'}^n, \quad q_{h'} = e^{2\pi i\tau/h'}$$

and all such expansions are equally plausible Fourier series of $f$ at $s$. In particular, when $k$ is odd the leading coefficient is only determined up to sign, and thinking of $f(s)$ as $a_0$ does not give it a well defined value. What is well defined is whether $a_0$ is 0, and so the intuition that a cusp form vanishes at all the cusps makes sense.

For more examples of modular forms with respect to congruence subgroups, start from the weight 2 Eisenstein series

$$G_2(\tau) = \sum_{c\in\mathbf{Z}} \sum_{d\in\mathbf{Z}'_c} \frac{1}{(c\tau+d)^2}$$

where $\mathbf{Z}'_c = \mathbf{Z} - \{0\}$ if $c = 0$ and $\mathbf{Z}'_c = \mathbf{Z}$ otherwise. This series converges only conditionally, but the terms are arranged so that in specializing equation (1.2) to $k = 2$, the ensuing calculation remains valid to give

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n, \quad q = e^{2\pi i\tau}, \ \sigma(n) = \sum_{\substack{d|n \\ d>0}} d$$

(Exercise 1.2.8(a)). Conditional convergence keeps $G_2$ from being weakly modular. Instead, a calculation that should leave the reader deeply appreciative of absolute convergence in the future shows that

$$(G_2[\gamma]_2)(\tau) = G_2(\tau) - \frac{2\pi i c}{c\tau+d} \quad \text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \tag{1.4}$$

(Exercise 1.2.8(b–c)). The corrected function $G_2(\tau) - \pi/\mathrm{Im}(\tau)$ is weight-2 invariant under $\mathrm{SL}_2(\mathbf{Z})$ (Exercise 1.2.8(d)), but it is not holomorphic. However, for any positive integer $N$, if

$$G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau)$$

then $G_{2,N} \in \mathcal{M}_2(\Gamma_0(N))$ (Exercise 1.2.8(e)). We will see many more Eisenstein series in Chapter 4.

Weight 2 Eisenstein series solve the four squares problem from the beginning of the section. The modular forms $G_{2,2}$ and $G_{2,4}$ work out to (Exercise 1.2.9)

$$G_{2,2}(\tau) = -\frac{\pi^2}{3}\Big(1 + 24\sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d|n \\ d \text{ odd}}} d\Big)q^n\Big)$$

and

$$G_{2,4}(\tau) = -\pi^2\Big(1 + 8\sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d|n \\ 4\nmid d}} d\Big)q^n\Big).$$

Now, $G_{2,2} \in \mathcal{M}_2(\Gamma_0(2)) \subset \mathcal{M}_2(\Gamma_0(4))$ (the smaller group allows more weight-2 invariant functions and the other conditions in Definition 1.2.3 make no reference to a congruence subgroup, so the smaller group has more modular forms) and $G_{2,4} \in \mathcal{M}_2(\Gamma_0(4))$. Exercise 3.9.3 will show that $\dim(\mathcal{M}_2(\Gamma_0(4))) = 2$, so $G_{2,2}$ and $G_{2,4}$, which visibly are linearly independent, are a basis. Recall that the function $\theta(\tau,4)$ also lies in the space $\mathcal{M}_2(\Gamma_0(4))$. Thus $\theta = aG_{2,2} + bG_{2,4}$ for some $a, b \in \mathbf{C}$, and the expansions

$$\begin{aligned}
\theta(\tau,4) &= 1 + 8q + \cdots, \\
-\frac{3}{\pi^2}G_{2,2}(\tau) &= 1 + 24q + \cdots, \\
-\frac{1}{\pi^2}G_{2,4}(\tau) &= 1 + 8q + \cdots,
\end{aligned}$$

show that $\theta(\tau,4) = -(1/\pi^2)G_{2,4}(\tau)$. Equating the Fourier coefficients gives the representation number of $n$ as a sum of four squares,

$$r(n,4) = 8\sum_{\substack{0<d|n \\ 4\nmid d}} d, \quad n \geq 1.$$

In particular, if $4 \nmid n$ then $r(n,4) = 8\sigma_1(n)$. The two squares problem, the six squares problem, and the eight squares problem are solved similarly once additional machinery is in place. For any even $s \geq 10$ the same methods give an asymptotic solution $\tilde{r}(n,s)$ to the $s$ squares problem, meaning that $\lim_{n\to\infty} \tilde{r}(n,s)/r(n,s) = 1$. Exercise 4.8.7 will discuss all of this.

For another application of weight 2 Eisenstein series, first normalize $G_2$ to

$$E_2(\tau) = \frac{G_2(\tau)}{2\zeta(2)} = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n.$$

Then equation (1.4) with $\gamma = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ specializes to

$$\tau^{-2}E_2(-1/\tau) = E_2(\tau) + \frac{12}{2\pi i\tau}. \tag{1.5}$$

The *Dedekind eta function* is the infinite product

$$\eta(\tau) = q_{24}\prod_{n=1}^{\infty}(1 - q^n), \quad q_{24} = e^{2\pi i\tau/24},\ q = e^{2\pi i\tau}.$$